EI Exponential Integral

EI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$.The function Exponential Integral (noted Ei) is defined by the following second order differential equation

$$(1-x)\frac{\partial y(x)}{\partial x} + x\frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{EI.1.1}$$

The initial conditions of EI.1.1 at 0 are not simple to state, since 0 is a (regular) singular point.

EI.2 Series and asymptotic expansions

EI.2.1 Asymptotic expansion at 0.

EI.2.1.1 First terms.

$$\mathrm{Ei}(x) \approx \left(-x - \frac{x^2}{4} - \frac{x^3}{18} - \frac{x^4}{96} - \frac{x^5}{600} - \frac{x^6}{4320} - \frac{x^7}{35280} - \frac{x^8}{322560} - \ln(x) + \gamma \ldots\right). \tag{EI.2.1.1}$$

EI.2.1.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

EI.2.2 Asymptotic expansion at ∞.

EI.2.2.1 First terms.

$$\mathrm{Ei}(x) \approx \mathrm{e}^{\frac{1}{x}}\, x y_0(x),$$

where

$$y_0(x) = 1 + x + 2x^2 + 6x^3 + 2\ldots$$

EI.2.2.2 General form.

EI.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$-u(n)n + u(n-1)\left(-1 + 2n + (n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = 1$$

This recurrence has the closed form solution

$$u(n) = \Gamma(n+1).$$