CI Cosine Integral

CI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$.The function Cosine Integral (noted Ci) is defined by the following third order differential equation

$$x \frac{\partial y(x)}{\partial x} + 2 \frac{\partial^2 y(x)}{\partial x^2} + x \frac{\partial^3 y(x)}{\partial x^3} = 0. \tag{CI.1.1}$$

The initial conditions of CI.1.1 at 0 are not simple to state, since 0 is a (regular) singular point.

Related function: Sine Integral

CI.2 Series and asymptotic expansions

CI.2.1 Asymptotic expansion at 0.

CI.2.1.1 First terms.

$$\mathrm{Ci}(x) \approx \left(\frac{x^2}{4} - \frac{x^4}{96} + \frac{x^6}{4320} - \frac{x^8}{322560} - \ln(x) + \gamma \ldots \right). \tag{CI.2.1.1.1}$$

CI.2.1.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

CI.2.2 Asymptotic expansion at ∞.

CI.2.2.1 First terms.

$$\mathrm{Ci}(x) \approx \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,1}(1+\xi^2)}{x} \right)} x y_0(x) + \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,2}(1+\xi^2)}{x} \right)} x y_1(x),$$

where

$$y_0(x) = \frac{i}{2} + \frac{i}{2}\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)x + \left(\frac{i}{4} + \frac{5i}{4}\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)^2\right)x^2 + \Bigg(i\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big) + 4\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)\left(\frac{i}{4} + \frac{5i}{4}\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)^2\right)\Bigg)x^3 + 2\ldots$$

$$y_1(x) = \frac{-i}{2} - \frac{i}{2}\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)x - -\left(\frac{-i}{4} - \frac{5i}{4}\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)^2\right)x^2 - -\Bigg(-i\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big) + 4\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)\left(\frac{-i}{4} - \frac{5i}{4}\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)^2\right)\Bigg)x^3 + 2\ldots$$

CI.2.2.2 General form.

CI.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$2u(n)n + u(n-1)\Big(-2\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big) - 5\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)(n-1) - 3(n-1)^2\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)\Big) + u(n-2)\big(8 - 5n - 4(n-2)^2 - (n-2)^3\big) = 0$$

whose initial conditions are given by

$$u(1) = \frac{i}{2}\operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)$$
$$u(0) = \frac{i}{2}$$

CI.2.2.2.2 Auxiliary function $y_1(x)$. The coefficients $u(n)$ of $y_1(x)$ satisfy the following recurrence

$$2u(n)n + u(n-1)\Big(-2\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big) - 5\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)(n-1) - 3(n-1)^2\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)\Big) + u(n-2)\big(8 - 5n - 4(n-2)^2 - (n-2)^3\big) = 0$$

whose initial conditions are given by

$$u(1) = -\frac{i}{2}\operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)$$
$$u(0) = \frac{-i}{2}$$