BSY Bessel Y

BSY.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$ and let ν denote a parameter (independent of x).The function Bessel Y (noted Y_ν) is defined by the following second order differential equation

$$\text{(BSY.1.1)} \qquad \left(x^2 - \nu^2\right) y(x) + x \frac{\partial y(x)}{\partial x} + x^2 \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

Although 0 is a singularity of BSY.1.1, the initial conditions can be given by

$$\text{(BSY.1.2)} \qquad \begin{aligned} \left[x^{(-\nu)}\right] \mathrm{Y}_\nu(x) &= -\frac{1}{\frac{\Gamma(-\nu+1)\sin(\nu\pi)}{2^\nu}}, \\ \left[x^{\nu}\right] \mathrm{Y}_\nu(x) &= \frac{\cos(\nu\pi)}{\Gamma(\nu+1) 2^\nu \sin(\nu\pi)}. \end{aligned}$$

The formulae of this document are valid for $-2\nu \notin \mathbb{Z}$.
Related functions: Hankel H1,Hankel H2,Bessel J

BSY.2 Series and asymptotic expansions

BSY.2.1 Asymptotic expansion at ∞.

BSY.2.1.1 First terms.

$$\begin{aligned} \mathrm{Y}_\nu(x) \approx \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,2}(1+\xi^2)}{x}\right)} \sqrt{x} y_0(x) + \\ \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,1}(1+\xi^2)}{x}\right)} \sqrt{x} y_1(x), \end{aligned}$$

where

$$y_0(x) = \frac{-i\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}}{2\sqrt{\pi}} + \frac{-i\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x}{16\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)} - \frac{i\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x^2}{256\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^2} + \frac{-i\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x^3}{6144\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^3} + 2\ldots$$

$$y_1(x) = \frac{i\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}}{2\sqrt{\pi}} - \frac{-i\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x}{16\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)} + \frac{i\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x^2}{256\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^2} - \frac{-i\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x^3}{6144\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^3} + 2\ldots$$

BSY.2.1.2 General form.

BSY.2.1.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$8u(n)n\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right) + u(n-1)\left(-4\nu^2-3+4n+4(n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \frac{-i\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}}{2\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n) = \left(-i2^{\left(n+\frac{1}{2}\right)}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^n\Gamma\left(\nu+\frac{1}{2}+n\right)\Gamma\left(n-\nu+\frac{1}{2}\right)\sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n(-1)^n\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\right)\Big/\left(28^n\pi^{\frac{3}{2}}\Gamma(n+1)\right).$$

BSY.2.1.2.2 Auxiliary function $y_1(x)$. The coefficients $u(n)$ of $y_1(x)$ satisfy the following recurrence

$$8u(n)n\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right) + u(n-1)\left(-4\nu^2-3+4n+4(n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \frac{i\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}}{2\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n) = \left(i2^{\left(n+\frac{1}{2}\right)} \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^n \Gamma\left(\nu+\frac{1}{2}+n\right)\Gamma\left(n-\nu+\frac{1}{2}\right) \sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n(-1)^n \, e^{\frac{i}{4}\pi(2\nu+1)}\right) \Bigg/ \left(28^n \pi^{\frac{3}{2}} \Gamma(n+1)\right).$$

BSY.2.2 Asymptotic expansion at 0**.**

BSY.2.2.1 First terms.

$$\mathrm{Y}_\nu(x) \approx \left(-\frac{1}{\frac{\Gamma(-\nu+1)\sin(\nu\pi)}{2^\nu}} - \frac{x^2}{\frac{4(\nu-1)\Gamma(-\nu+1)\sin(\nu\pi)}{2^\nu}} - \frac{x^4}{\frac{32(\nu-1)(\nu-2)\Gamma(-\nu+1)\sin(\nu\pi)}{2^\nu}} - \frac{x^6}{\frac{384(\nu-1)(\nu-2)(\nu-3)\Gamma(-\nu+1)\sin(\nu\pi)}{2^\nu}} - \frac{x^8}{\frac{6144(\nu-1)(\nu-2)(\nu-3)(\nu-4)\Gamma(-\nu+1)\sin(\nu\pi)}{2^\nu}} \cdots\right) \Bigg/ x^\nu + x^\nu\left(\frac{\cos(\nu\pi)}{\Gamma(\nu+1)2^\nu\sin(\nu\pi)} - \frac{x^2\cos(\nu\pi)}{4(\nu+1)\Gamma(\nu+1)2^\nu\sin(\nu\pi)} + \frac{x^4\cos(\nu\pi)}{32(\nu+1)(\nu+2)\Gamma(\nu+1)2^\nu\sin(\nu\pi)} - \frac{x^6\cos(\nu\pi)}{384(\nu+1)(\nu+2)(\nu+3)\Gamma(\nu+1)2^\nu\sin(\nu\pi)} + \frac{x^8\cos(\nu\pi)}{6144(\nu+1)(\nu+2)(\nu+3)(\nu+4)\Gamma(\nu+1)2^\nu\sin(\nu\pi)} \cdots\right). \tag{BSY.2.2.1.1}$$

BSY.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).