BSK Bessel K

BSK.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$ and let ν denote a parameter (independent of x).The function Bessel K (noted K_ν) is defined by the following second order differential equation

$$-x^2 + \nu^2 y(x) + x\frac{\partial y(x)}{\partial x} + x^2 \frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{BSK.1.1}$$

Although 0 is a singularity of BSK.1.1, the initial conditions can be given by

$$\begin{aligned} \left[x^{(-\nu)}\right] \mathrm{K}_\nu(x) &= 2^{(\nu-1)}\Gamma(\nu), \\ \left[x^{\nu}\right] \mathrm{K}_\nu(x) &= -\frac{\pi}{\Gamma(\mu+1)\sin(\mu\pi)2^{(\mu+1)}}. \end{aligned} \tag{BSK.1.2}$$

The formulae of this document are valid for $2\nu \notin \mathbb{Z}$.

Related function: Bessel I

BSK.2 Series and asymptotic expansions

BSK.2.1 Asymptotic expansion at ∞.

BSK.2.1.1 First terms.

$$\mathrm{K}_\nu(x) \approx \mathrm{e}^{\left(-\frac{1}{x}\right)} \sqrt{x} y_0(x),$$

where

$$\begin{aligned} y_0(x) = {} & \frac{\sqrt{2}\sqrt{\pi}}{2} - \frac{-\left(4\nu^2-1\right)\sqrt{2}\sqrt{\pi}x}{16} + \\ & \frac{\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\sqrt{\pi}x^2}{256} - \\ & \frac{-\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\sqrt{\pi}x^3}{6144} + 2\ldots \end{aligned}$$

BSK.2.1.2 General form.

BSK.2.1.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$8u(n)n + u(n-1)\left(-4\nu^2 - 3 + 4n + 4(n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \frac{\sqrt{2}\sqrt{\pi}}{2}$$

This recurrence has the closed form solution

$$u(n) = \frac{2^{\left(n+\frac{1}{2}\right)}\Gamma\left(n+\nu+\frac{1}{2}\right)\Gamma\left(n-\nu+\frac{1}{2}\right)\sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n}{28^n\sqrt{\pi}\Gamma(n+1)}.$$

BSK.2.2 Asymptotic expansion at 0.

BSK.2.2.1 First terms.

$$\begin{aligned} \mathrm{K}_\nu(x) \approx & \left(2^{(\nu-1)}\Gamma(\nu) - \frac{x^2 2^{(\nu-1)}\Gamma(\nu)}{4\nu-4} + \frac{x^4 2^{(\nu-1)}\Gamma(\nu)}{32(\nu-1)(\nu-2)} - \right. \\ & \left. \frac{x^6 2^{(\nu-1)}\Gamma(\nu)}{384(\nu-1)(\nu-2)(\nu-3)} + \frac{x^8 2^{(\nu-1)}\Gamma(\nu)}{6144(\nu-1)(\nu-2)(\nu-3)(\nu-4)} \cdots\right) \Big/ \\ & x^\nu + x^\nu\left(-\frac{\pi}{\Gamma(\mu+1)\sin(\mu\pi)2^{(\mu+1)}} - \right. \\ & \frac{x^2\pi}{4(\nu+1)\Gamma(\mu+1)\sin(\mu\pi)2^{(\mu+1)}} - \\ & \frac{x^4\pi}{32(\nu+1)(\nu+2)\Gamma(\mu+1)\sin(\mu\pi)2^{(\mu+1)}} - \\ & \frac{x^6\pi}{384(\nu+1)(\nu+2)(\nu+3)\Gamma(\mu+1)\sin(\mu\pi)2^{(\mu+1)}} - \\ & \left. \frac{x^8\pi}{6144(\nu+1)(\nu+2)(\nu+3)(\nu+4)\Gamma(\mu+1)\sin(\mu\pi)2^{(\mu+1)}} \cdots\right). \end{aligned} \tag{BSK.2.2.1.1}$$

BSK.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).