BSJ Bessel J

BSJ.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$ and let ν denote a parameter (independent of x).The function Bessel J (noted J_ν) is defined by the following second order differential equation

$$(x^2 - \nu^2)y(x) + x\frac{\partial y(x)}{\partial x} + x^2\frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{BSJ.1.1}$$

Although 0 is a singularity of BSJ.1.1, the initial conditions can be given by

$$\frac{\partial \frac{\mathrm{J}_\nu(x)}{x^\nu}}{\partial x} = \frac{1}{\Gamma(\nu+1)2^\nu}. \tag{BSJ.1.2}$$

The formulae of this document are valid for $-2\nu \notin \mathbb{Z}$.

Related functions: Hankel H1,Hankel H2,Bessel Y

BSJ.2 Series and asymptotic expansions

BSJ.2.1 Asymptotic expansion at 0.

BSJ.2.1.1 First terms.

$$\begin{aligned}\mathrm{J}_\nu(x) \approx x^\nu \Bigg(&\frac{1}{\Gamma(\nu+1)2^\nu} - \frac{x^2}{4\Gamma(\nu+1)2^\nu(\nu+1)} + \\ &\frac{x^4}{32\Gamma(\nu+1)2^\nu(\nu+1)(\nu+2)} - \\ &\frac{x^6}{384\Gamma(\nu+1)2^\nu(\nu+1)(\nu+2)(\nu+3)} \cdots \Bigg).\end{aligned} \tag{BSJ.2.1.1.1}$$

BSJ.2.1.2 General form.

$$\mathrm{J}_\nu(x) \approx x^\nu \sum_{n=0}^{\infty} u(n)x^n. \tag{BSJ.2.1.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

$$u(n)\big(-\nu^2 + (\nu+n)^2\big) + u(n-2) = 0. \tag{BSJ.2.1.2.2}$$

Initial conditions of BSJ.2.1.2.2 are given by

$$\begin{aligned} u(1) &= 0, \\ u(0) &= \frac{1}{\Gamma(\nu+1)2^\nu}. \end{aligned} \tag{BSJ.2.1.2.3}$$

The recurrence BSJ.2.1.2.2 has the closed form solution

$$u(2n+1) = 0,$$
(BSJ.2.1.2.4)
$$u(2n) = \frac{(-1)^n}{2^\nu 4^n \Gamma(n+1)\Gamma(n+\nu+1)}.$$

BSJ.2.2 Asymptotic expansion at ∞.

BSJ.2.2.1 First terms.

$$\mathrm{J}_\nu(x) \approx \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,2}(1+\xi^2)}{x}\right)} \sqrt{x} y_0(x) + \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,1}(1+\xi^2)}{x}\right)} \sqrt{x} y_1(x),$$

where

$$y_0(x) = \frac{\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}}{2\sqrt{\pi}} - \frac{-\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x}{16\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)} + \frac{\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x^2}{256\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^2} - \frac{-\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x^3}{6144\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^3} + 2\ldots$$

$$y_1(x) = \frac{\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}}{2\sqrt{\pi}} - \frac{-\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x}{16\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)} + \frac{\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x^2}{256\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^2} - \frac{-\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x^3}{6144\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^3} + 2\ldots$$

BSJ.2.2.2 General form.

BSJ.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$8u(n)n\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right) + u(n-1)\left(-4\nu^2-3+4n+4(n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \frac{\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}}{2\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n) = \left(2^{\left(n+\frac{1}{2}\right)} \operatorname{RootOf}_{\xi,2}\big(1+\xi^2\big)^n \Gamma\left(n-\nu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}+n\right) \sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n(-1)^n \, \mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\right) \Big/ \left(2\Gamma(n+1)8^n\pi^{\frac{3}{2}}\right).$$

BSJ.2.2.2.2 Auxiliary function $y_1(x)$. The coefficients $u(n)$ of $y_1(x)$ satisfy the following recurrence

$$8u(n)n \operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big) + u(n-1)\big(-4\nu^2 - 3 + 4n + 4(n-1)^2\big) = 0$$

whose initial conditions are given by

$$u(0) = \frac{\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}}{2\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n) = \left(2^{\left(n+\frac{1}{2}\right)} \operatorname{RootOf}_{\xi,1}\big(1+\xi^2\big)^n \Gamma\left(n-\nu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}+n\right) \sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n(-1)^n \, \mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\right) \Big/ \left(2\Gamma(n+1)8^n\pi^{\frac{3}{2}}\right).$$