BSI Bessel I

BSI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$ and let ν denote a parameter (independent of x).The function Bessel I (noted I_ν) is defined by the following second order differential equation

(BSI.1.1) $$-x^2 + \nu^2 y(x) + x \frac{\partial y(x)}{\partial x} + x^2 \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

Although 0 is a singularity of BSI.1.1, the initial conditions can be given by

(BSI.1.2) $$\frac{\partial \frac{\mathrm{I}_\nu(x)}{x^\nu}}{\partial x} = \frac{1}{\Gamma(\nu+1)2^\nu}.$$

The formulae of this document are valid for $-2\nu \notin \mathbb{Z}$.

Related function: Bessel K

BSI.2 Series and asymptotic expansions

BSI.2.1 Asymptotic expansion at ∞.

BSI.2.1.1 First terms.

$$\mathrm{I}_\nu(x) \approx \mathrm{e}^{\left(-\frac{1}{x}\right)} \sqrt{x} y_0(x),$$

where

$$y_0(x) = \frac{\sqrt{2}}{2\sqrt{\pi}} - \frac{-\left(4\nu^2-1\right)\sqrt{2}x}{16\sqrt{\pi}} + \frac{\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}x^2}{256\sqrt{\pi}} - \frac{-\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}x^3}{6144\sqrt{\pi}} + 2\ldots$$

BSI.2.1.2 General form.

BSI.2.1.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$8u(n)n + u(n-1)\left(-4\nu^2 - 3 + 4n + 4(n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \frac{\sqrt{2}}{2\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n) = \frac{2^{\left(n+\frac{1}{2}\right)}\Gamma\left(n+\nu+\frac{1}{2}\right)\Gamma\left(n-\nu+\frac{1}{2}\right)\sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n}{28^n \pi^{\frac{3}{2}}\Gamma(n+1)}.$$

BSI.2.2 Asymptotic expansion at 0.

BSI.2.2.1 First terms.

$$\mathrm{I}_{\nu}(x) \approx x^{\nu}\left(\frac{1}{\Gamma(\nu+1)2^{\nu}} + \frac{x^2}{4\Gamma(\nu+1)2^{\nu}(\nu+1)} + \frac{x^4}{32\Gamma(\nu+1)2^{\nu}(\nu+1)(\nu+2)} + \frac{x^6}{384\Gamma(\nu+1)2^{\nu}(\nu+1)(\nu+2)(\nu+3)} \cdots\right). \tag{BSI.2.2.1.1}$$

BSI.2.2.2 General form.

$$\mathrm{I}_{\nu}(x) \approx x^{\nu}\sum_{n=0}^{\infty} u(n)x^n. \tag{BSI.2.2.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

$$u(n)\bigl(-\nu^2 + (\nu+n)^2\bigr) - u(n-2) = 0. \tag{BSI.2.2.2.2}$$

Initial conditions of BSI.2.2.2.2 are given by

$$u(0) = \frac{1}{\Gamma(\nu+1)2^{\nu}}, \quad u(1) = 0. \tag{BSI.2.2.2.3}$$

The recurrence BSI.2.2.2.2 has the closed form solution

$$u(2n) = \frac{1}{2^{\nu}4^{n}\Gamma(n+1)\Gamma(n+\nu+1)}, \quad u(2n+1) = 0. \tag{BSI.2.2.2.4}$$