BI Airy Bi

BI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$.The function Airy Bi (noted Bi) is defined by the following second order differential equation

(BI.1.1) $$-xy(x) + \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

The initial conditions of BI.1.1 are given at 0 by

(BI.1.2) $$\mathrm{Bi}(0) = \frac{3^{\frac{5}{6}}}{3\Gamma\left(\frac{2}{3}\right)}, \qquad \frac{\partial\,\mathrm{Bi}(x)}{\partial x}(0) = \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{2\pi}.$$

Related function: Airy Ai

BI.2 Series and asymptotic expansions

BI.2.1 Taylor expansion at 0.

BI.2.1.1 First terms.

(BI.2.1.1.1) $$\mathrm{Bi}(x) = \frac{3^{\frac{5}{6}}}{3\Gamma\left(\frac{2}{3}\right)} + \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{2\pi}x + \frac{3^{\frac{5}{6}}}{18\Gamma\left(\frac{2}{3}\right)}x^3 + \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{24\pi}x^4 + \frac{3^{\frac{5}{6}}}{540\Gamma\left(\frac{2}{3}\right)}x^6 + \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{1008\pi}x^7 + \frac{3^{\frac{5}{6}}}{38880\Gamma\left(\frac{2}{3}\right)}x^9 + \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{90720\pi}x^{10} + \frac{3^{\frac{5}{6}}}{5132160\Gamma\left(\frac{2}{3}\right)}x^{12} + \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{14152320\pi}x^{13} + \frac{3^{\frac{5}{6}}}{1077753600\Gamma\left(\frac{2}{3}\right)}x^{15} + \mathrm{O}\left(x^{16}\right).$$

BI.2.1.2 General form.

(BI.2.1.2.1) $$\mathrm{Bi}(x) = \sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(BI.2.1.2.2) $$-u(n) + \left(n^2 + 5n + 6\right)u(n+3) = 0.$$

Initial conditions of BI.2.1.2.2 are given by

(BI.2.1.2.3)
$$u(0) = \frac{3^{\frac{5}{6}}}{3\Gamma\left(\frac{2}{3}\right)}, \quad u(1) = \frac{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}{2\pi}, \quad u(2) = 0.$$

The recurrence BI.2.1.2.2 has the closed form solution

(BI.2.1.2.4)
$$u(3n+2) = 0, \quad u(3n) = \frac{3^{\left(\frac{5}{6}-2n\right)}}{3\Gamma(n+1)\Gamma\left(n+\frac{2}{3}\right)}, \quad u(3n+1) = \frac{3^{\left(\frac{1}{6}-2n\right)}}{3\Gamma\left(n+\frac{4}{3}\right)\Gamma(n+1)}.$$

BI.2.2 Asymptotic expansion at ∞.

BI.2.2.1 First terms.

$$\mathrm{Bi}(x) \approx \mathrm{e}^{\left(\frac{-2}{3\xi^3}\right)} \sqrt{\xi}\left(\frac{-i}{\sqrt{\pi}} + \frac{5i\xi^3}{48\sqrt{\pi}} + \dots\right)$$

where $\xi = -\sqrt{\frac{1}{x}}$

BI.2.2.2 General form.

$$\mathrm{Bi}(x) \approx \mathrm{e}^{\left(\frac{-2}{3\xi^3}\right)} \sqrt{\xi}\sum_{n=0}^{\infty} u(n)\xi^n$$

where $\xi = -\sqrt{\frac{1}{x}}$The coefficients $u(n)$ satisfy the following recurrence

$$16u(n)n + u(n-3)\left(-31+12n+4(n-3)^2\right) = 0.$$

whose initial conditions are given by

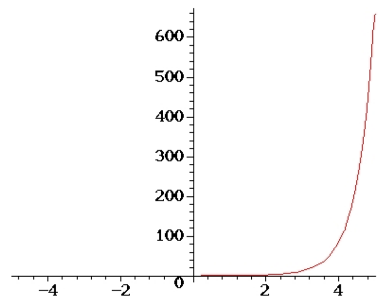
$$u(2) = 0, \quad u(1) = 0, \quad u(0) = \frac{-i}{\sqrt{\pi}}.$$

This recurrence has the closed form solution

$$u(3n+2) = 0, \quad u(3n+1) = 0, \quad u(3n) = \frac{-i(-1)^n 6^{(2n)}\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{2\pi^{\frac{3}{2}}48^n\Gamma(n+1)}.$$

BI.3 Graphs

BI.3.1 Real axis.



BI.3.2 Complex plane.