AHCS Inverse Hyperbolic Cosecant

AHCS.1 Introduction

Let x be a complex variable of $\mathbb{C}\backslash\{0\}$.The function Inverse Hyperbolic Cosecant (noted arccsch) is defined by the following second order differential equation

$$\left(2x^2+1\right)\frac{\partial y(x)}{\partial x}+\left(x^3+x\right)\frac{\partial^2 y(x)}{\partial x^2}=0. \tag{AHCS.1.1}$$

The initial conditions of AHCS.1.1 at 0 are not simple to state, since 0 is a (regular) singular point.

AHCS.2 Series and asymptotic expansions

AHCS.2.1 Asymptotic expansion at $-i$.

AHCS.2.1.1 First terms.

$$\operatorname{arccsch}(x) \approx \left(\frac{i}{2}\pi \ldots\right) + \sqrt{x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)}\left(1 - i + \left(\frac{5}{12} - \frac{5i}{12}\right)\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right) + (1-i)\left(\frac{3}{40} + \frac{11\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2}{32}\right)\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^2 + (1-i)\left(\frac{25\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)}{336} + \frac{17\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(12+55\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\right)}{2688}\right)\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^3 + (1-i)\left(\frac{7}{384} + \frac{385\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2}{4608} + \frac{23\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\left(404+935\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\right)}{55296}\right)\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^4 + (1-i)\left(\frac{15\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)}{704} + \frac{51\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(12+55\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\right)}{28160} + 29\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(1008 + 13912\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2 + 21505\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^4\right)\Big/1351680\right)\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^5 + (1-i)\left(\frac{77}{13312} + \frac{4235\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2}{159744} + \frac{253\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\left(404+935\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\right)}{1916928} + 7\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\left(87408 + 538088\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2 + 623645\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^4\right)\Big/7667712\right)\left(x - \operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\right)^6 + (1-i)\left(\frac{13\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)}{1792} + \frac{221\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(12+55\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2\right)}{358400} + 377\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(1008 + 13912\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2 + 21505\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^4\right)\Big/51609600 + 41\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)\Big(44352 + 1223984\operatorname{RootOf}_{\xi,2}\left(1+\xi^2\right)^2 + \tag{AHCS.2.1.1.1}$$

AHCS.2.1.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

AHCS.2.2 Asymptotic expansion at i.

AHCS.2.2.1 First terms.

$$\operatorname{arccsch}(x) \approx \left(-\frac{i}{2}\pi\ldots\right) + \sqrt{x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)}\Bigg(1 + i + \left(\frac{5}{12} + \frac{5i}{12}\right)\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\left(x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\right) + (1+i)\left(\frac{3}{40} + \frac{11\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2}{32}\right)\left(x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^2 + (1+i)\Bigg(\frac{25\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)}{336} + \frac{17\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\left(12+55\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\right)}{2688}\Bigg)\left(x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^3 + (1+i)\Bigg(\frac{7}{384} + \frac{385\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2}{4608} + \frac{23\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\left(404+935\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\right)}{55296}\Bigg)\left(x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^4 + (1+i)\Bigg(\frac{15\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)}{704} + \frac{51\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\left(12+55\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\right)}{28160} + 29\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\Big(1008 + 13912\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2 + 21505\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^4\Big)\Big/1351680\Bigg)\left(x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^5 + (1+i)\Bigg(\frac{77}{13312} + \frac{4235\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2}{159744} + \frac{253\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\left(404+935\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\right)}{1916928} + 7\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\Big(87408 + 538088\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2 + 623645\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^4\Big)\Big/7667712\Bigg)\left(x - \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\right)^6 + (1+i)\Bigg(\frac{13\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)}{1792} + \frac{221\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\left(12+55\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2\right)}{358400} + 377\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\Big(1008 + 13912\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2 + 21505\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^4\Big)\Big/51609600 + 41\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)\Big(44352 + 1223984\operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^2 + \tag{AHCS.2.2.1.1}$$

AHCS.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

AHCS.2.3 Asymptotic expansion at 0**.**

AHCS.2.3.1 First terms.

$$\operatorname{arctanh}(x) \approx \left(\ln(2) - \frac{x^2}{4} + \frac{3x^4}{32} - \frac{5x^6}{96} + \frac{35x^8}{1024} + \ln(x)\dots\right). \tag{AHCS.2.3.1}$$

AHCS.2.3.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).