Poisson-Voronoi Tessellations

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The d**-dimensional Poisson process** of **intensity** λ is a random scattering of points (called **particles**) in $\mathbb{R}^d$ that meets the following two requirements. Let $S \subseteq \mathbb{R}^d$ denote a measurable set of finite volume μ and $N(S)$ denote the number of particles falling in S. We have [1, 2]

- $\mathrm{P}\{N(S) = n\} = e^{-\lambda\mu}(\lambda\mu)^n/n!$ for any S, for any $n = 0, 1, 2, ...$, and
- if $S_1, \ldots, S_k$ are disjoint measurable sets, then $N(S_1), \ldots, N(S_k)$ are independent random variables.

In particular, the location of S in $\mathbb{R}^d$ is immaterial (stationarity) and $\mathrm{E}(N(S)) = \lambda\mu = \mathrm{Var}(N(S))$ (equality of mean and variance). An alternative characterization of the Poisson process involves the limit of the uniform distribution on expanding cubes $C \subseteq \mathbb{R}^d$. Let ν denote the volume of C. Given m independent uniformly distributed particles in C and a measurable set $S \subseteq C$ of volume μ, the probability that exactly n particles fall in S is

$$\frac{m!}{n!(m-n)!}\left(\frac{\mu}{\nu}\right)^n\left(1-\frac{\mu}{\nu}\right)^{m-n} \to e^{-\lambda\mu}\frac{(\lambda\mu)^n}{n!},$$

which occurs in the limit as $\nu \to \infty$ in such a way that $m/\nu \to \lambda$. The interpretation of λ as a rate or intensity is thus clear, as is the phrase *binomial process* to denote a Uniform (C) distribution.

Here is a sample problem involving the Poisson process; assume for simplicity henceforth that $\lambda = 1$. Let ξ be an arbitrary point in $\mathbb{R}^d$ and R denote the distance from ξ to its nearest neighboring particle. What can be said about R? If $\omega_d = \pi^{d/2}\Gamma(d/2+1)^{-1}$ is the volume of the unit d-ball, then [3, 4, 5]

$$\mathrm{P}\{R > r\} = \mathrm{P}\{d\text{-ball of radius } r \text{ contains no particles}\} = e^{-\omega_d r^d}$$

which implies that

$$\begin{aligned} \mathrm{E}(R) &= \omega_d^{-1/d}\Gamma\left(\frac{1}{d}+1\right) = \begin{cases} \dfrac{1}{2} & \text{if } d=1 \text{ or } 2, \\ \left(\dfrac{3}{4\pi}\right)^{1/3}\Gamma\left(\dfrac{4}{3}\right) & \text{if } d=3 \end{cases} \\ &= \begin{cases} 0.5 & \text{if } d=1 \text{ or } 2, \\ 0.5539602783... & \text{if } d=3. \end{cases} \end{aligned}$$

Likewise,

$$\mathrm{E}(R^2) = \omega_d^{-2/d}\Gamma\left(\frac{2}{d}+1\right) = \begin{cases} \dfrac{1}{2} & \text{if } d=1, \\ \dfrac{1}{\pi} & \text{if } d=2, \\ \left(\dfrac{3}{4\pi}\right)^{2/3}\Gamma\left(\dfrac{5}{3}\right) & \text{if } d=3 \end{cases}$$

and thus

$$\mathrm{Var}(R) = \mathrm{E}(R^2) - \mathrm{E}(R)^2 = \begin{cases} 0.25 & \text{if } d=1, \\ 0.0683098861... & \text{if } d=2, \\ 0.0405357524... & \text{if } d=3. \end{cases}$$

We will consider a vastly more difficult version of this problem shortly. Of all unit-intensity scattering methods, the Poisson process is the "most random"; hence the forthcoming constants deserve to be better understood!

0.1. Cellular Parameters. Given any set of distinct particles $\{p_i\}_{i=1}^{\infty}$ in $\mathbb{R}^d$, the corresponding **Voronoi tessellation** is the subdivision of $\mathbb{R}^d$ into convex polyhedral cells $\{\Pi_i\}_{i=1}^{\infty}$ with the property that Π_i contains all points in $\mathbb{R}^d$ closer to p_i than to any other p_j, $j \neq i$. If $d = 1$, the cells are subintervals of the line characterized simply by length. If $d \geq 2$, the geometry is more elaborate. Our interest is in the scenario when the particles are realizations of a Poisson process of intensity 1; hence the cellular parameters are random variables. Applications of this material include any field involving pattern analysis: astronomy, geography, metallurgy, biology and socio-economic planning, to mention only a few [6, 7].

If $d = 1$ and M denotes the length of a typical cell, then $\mathrm{E}(M) = 1$ and $\mathrm{Var}(M) = 1/2$ [8]. If $d = 2$ or 3, the associated mean values are known exactly [8], but the derivation of second moment integrals is notoriously difficult. A closed-form expression has not been found for any of these integrals.

For the following, define expressions [9, 10]

$$\begin{aligned} f_V(x,y) &= 4((\pi/2+x)(1+2\sin(x)^2)+3\sin(x)\cos(x))\sec(x)^5 \times \\ &\qquad ((\pi/2+y)(1+2\sin(y)^2)+3\sin(y)\cos(y))\sec(y)^5, \end{aligned}$$

$$f_L(x,y) = ((\pi/2+x)\tan(x)+1)\sec(x)^2((\pi/2+y)\tan(y)+1)\sec(y)^2,$$
$$f_P(x,y) = (1+\sin(x))\sec(x)^4(1+\sin(y))\sec(y)^4,$$
$$f_M(x,y) = \sec(x)^3\sec(y)^3,$$
$$g(x,y) = (\pi/2+x+\sin(x)\cos(x))\sec(x)^2 + (\pi/2+y+\sin(y)\cos(y))\sec(y)^2,$$
$$h(\rho,\theta) = \rho^2\left(\pi-\theta+\sin(2\theta)/2\right) + (1+\rho^2-2\rho\cos(\theta))\left(\pi-\kappa(\rho,\theta)+\sin(2\kappa(\rho,\theta)/2\right)$$

where
$$\kappa(\rho,\theta) = \arccos\left(\frac{1-\rho\cos(\theta)}{\sqrt{1+\rho^2-2\rho\cos(\theta)}}\right).$$

A geometric interpretation of $h(\rho,\theta)$ is as the area of the union of two overlapping planar disks with unit distance between their centers, one with radius ρ and the other with radius $\sqrt{1+\rho^2-2\rho\cos(\theta)}$. When $d=2$, we have [10, 11, 12, 13]

$$\mathrm{E}(V) = 6,$$

$$\mathrm{E}(V^2) = 12\pi\int_{-\pi/2}^{\pi/2}\int_{-x}^{\pi/2} f_V(x,y)g(x,y)^{-4}\sin(x+y)\,dy\,dx + 18,$$

$$\mathrm{Var}(V) = 1.7808116990... = 37.7808116990... - \mathrm{E}(V)^2$$

where V is the number of vertices of the cell; [10, 11]

$$\mathrm{E}(L) = 5\pi^{3/2}\int_{-\pi/2}^{\pi/2}\int_{-x}^{\pi/2} f_L(x,y)g(x,y)^{-7/2}\left(\tan(x)+\tan(y)\right)dy\,dx = \frac{2}{3},$$

$$\mathrm{E}(L^2) = 16\pi\int_{-\pi/2}^{\pi/2}\int_{-x}^{\pi/2} f_L(x,y)g(x,y)^{-4}\left(\tan(x)+\tan(y)\right)^2 dy\,dx,$$

$$\mathrm{Var}(L) = 0.1856273347... = 0.6300717791... - \mathrm{E}(L)^2$$

where L is the length of an arbitrary edge;

$$\mathrm{E}(P) = 4,$$

$$\mathrm{E}(P^2) = 64\pi\int_{-\pi/2}^{\pi/2}\int_{-x}^{\pi/2} f_P(x,y)g(x,y)^{-3}\sin(x+y)\,dy\,dx + 6\,\mathrm{E}(L^2),$$

$$\mathrm{Var}(P) = 0.9454930107... = 16.9454930107... - \mathrm{E}(P)^2$$

where $P = \sum L$ is the total perimeter; and [9, 10, 11, 14, 15, 16]

$$\mathrm{E}(M) = 1,$$

$$\begin{aligned}\mathrm{E}(M^2) &= 2\pi \int_{-\pi/2}^{\pi/2} \int_{-x}^{\pi/2} f_M(x,y) g(x,y)^{-2} \sin(x+y)\, dy\, dx \\ &= 2\pi \int_0^{\infty} \int_0^{\pi} \rho\, h(\rho,\theta)^{-2}\, d\theta\, d\rho,\end{aligned}$$

$$\mathrm{Var}(M) = 0.2801760409... = 1.2801760409... - \mathrm{E}(M)^2$$

where M is the area of the cell. It is also known that $\mathrm{E}(M^3) = 1.999...$ [15].

For the following, define expressions [9, 17]

$$f_L(x,y) = \sec(x)^2(\sec(x) + \tan(x))^2 \sec(y)^2(\sec(y) + \tan(y))^2,$$

$$g(x,y) = \sec(x)^3(2/3 + \sin(x) - \sin(x)^3/3) + \sec(y)^3(2/3 + \sin(y) - \sin(y)^3/3),$$

$$\begin{aligned}h(\rho,\theta) &= \pi\rho^3\,(2/3 + 3\cos(\theta)/4 - \cos(3\theta)/12) + \pi(1 + \rho^2 - 2\rho\cos(\theta))^{3/2} \times \\ &\quad (2/3 + 3\cos(\kappa(\rho,\theta))/4 - \cos(3\kappa(\rho,\theta))/12)\end{aligned}$$

and $\kappa(\rho,\theta)$ is as before. A geometric interpretation of $h(\rho,\theta)$ as the volume of the union of two spatial balls again holds. When $d = 3$, we have [17]

$$\mathrm{E}(W) = \frac{144\pi^2}{24\pi^2 + 35} = 5.2275734378...,$$

$$\mathrm{Var}(W) = 2.4846406759... = 29.8121647244 - \mathrm{E}(W)^2$$

where W is the number of vertices of an arbitrary face of the cell; [12, 13, 17]

$$\mathrm{E}(V) = \frac{96\pi^2}{35} = 27.0709149287...,$$

$$\mathrm{Var}(V) = 44.4983886849... = 777.3328237620 - \mathrm{E}(V)^2$$

where $V = \sum W$ is the total number of vertices; [17]

$$\mathrm{E}(E) = \frac{144\pi^2}{35} = 40.6063723930...,$$

$$\mathrm{Var}(E) = 100.1213745412... = 1748.9988534645... - \mathrm{E}(E)^2$$

where $E = 3V/2$ is the number of edges;

$$\mathrm{E}(F) = \frac{48\pi^2}{35} + 2 = 15.5354574643...,$$

$$\mathrm{Var}(F) = 11.1245971712... = 252.4750357979... - \mathrm{E}(F)^2$$

where $F = V/2 + 2$ is the number of faces;

$$\begin{aligned}\mathrm{E}(L) &= \frac{35}{36\pi^{1/3}}\Gamma\left(\frac{13}{3}\right)\int_{-\pi/2}^{\pi/2}\int_{-x}^{\pi/2} f_L(x,y)\, g(x,y)^{-13/3}\left(\tan(x)+\tan(y)\right) dy\, dx \\ &= \frac{7}{9}\left(\frac{3}{4\pi}\right)^{1/3}\Gamma\left(\frac{4}{3}\right) = 0.4308579942...,\end{aligned}$$

$$\mathrm{E}(L^2) = \frac{35}{36\pi^{2/3}}\Gamma\left(\frac{14}{3}\right)\int_{-\pi/2}^{\pi/2}\int_{-x}^{\pi/2} f_L(x,y)\, g(x,y)^{-14/3}\left(\tan(x)+\tan(y)\right)^2 dy\, dx,$$

$$\mathrm{Var}(L) = 0.1052391356... = 0.2908777468... - \mathrm{E}(L)^2$$

where L is the length of an arbitrary edge;

$$\mathrm{E}(Q) = \frac{21}{24\pi^2+35}\left(\frac{4\pi}{3}\right)^{5/3}\Gamma\left(\frac{1}{3}\right) = 2.2523418064...,$$

$$\mathrm{Var}(Q) = 1.4699757822... = 6.5430193952... - \mathrm{E}(Q)^2$$

where Q is the perimeter of an arbitrary face;

$$\mathrm{E}(P) = \frac{3}{5}\left(\frac{4\pi}{3}\right)^{5/3}\Gamma\left(\frac{1}{3}\right) = 17.4955801644...,$$

$$\mathrm{Var}(P) = 13.6179400522... = 319.7132653418... - \mathrm{E}(P)^2$$

where $P = \sum L = \sum Q/2$ is the total perimeter;

$$\mathrm{E}(B) = \frac{35}{24\pi^2+35}\left(\frac{256\pi}{81}\right)^{1/3}\Gamma\left(\frac{2}{3}\right) = 0.3746830505...,$$

$$\mathrm{Var}(B) = 0.1423896695... = 0.2827770579 - \mathrm{E}(B)^2$$

where B is the surface area of an arbitrary face;

$$\mathrm{E}(A) = \left(\frac{256\pi}{3}\right)^{1/3}\Gamma\left(\frac{5}{3}\right) = 5.8208725950...,$$

$$\mathrm{Var}(A) = 2.1914834552... = 36.0740412231 - \mathrm{E}(A)^2$$

where $A = \sum B$ is the total surface area; and [9, 14, 16, 17]

$$\mathrm{E}(M) = 1,$$

$$\mathrm{E}(M^2) = \frac{8\pi^2}{3} \int_0^\infty \int_0^\pi \rho^2 \sin(\theta)\, h(\rho,\theta)^{-2}\, d\theta\, d\rho,$$

$$\mathrm{Var}(M) = 0.1790324378... = 1.1790324378... - \mathrm{E}(M)^2$$

where M is the volume of the cell.

0.2. Vertex Counts. Thus far we have discussed only moments of distributions associated with Poisson-Voronoi cells. The computation of actual probabilities seems to be hard. If $d = 2$, for example, what is the probability that an arbitrary cell is a triangle? The solution can be expressed as a complicated quadruple integral and turns out numerically to be [18, 19, 20]

$$\mathrm{P}(V = 3) = 0.01124001....$$

Simulation can be used to verify this result and the preceding moment estimates as well [21, 22, 23, 24, 25, 26, 27, 28, 29, 30]; for example, it appears that $\mathrm{P}(V = 4) = 0.1608...$ and $\mathrm{P}(V = 5) = 0.2594...$ [31]. Integral formulas for these latter probabilities [31, 32] evidently require further simplification to be numerically feasible. The function $\mathrm{P}(V = n)$ is apparently maximized when $n = 6$ and falls off for $n \geq 7$; it is known that asymptotically [33, 34]

$$\mathrm{P}(V = n) = \frac{C}{4\pi^2} \frac{(8\pi^2)^n}{(2n)!} \left(1 + O(n^{-1})\right)$$

as $n \to \infty$, where

$$\begin{aligned} C &= \prod_{j=1}^{\infty} \left(1 - \frac{1}{j^2} + \frac{4}{j^4}\right)^{-1} = 0.3443473089... \\ &= 4 \cdot \left|\Gamma\left(\tfrac{\sqrt{5}}{2} + i\tfrac{\sqrt{3}}{2}\right)\right|^2 \cdot \left|\Gamma\left(-\tfrac{\sqrt{5}}{2} - i\tfrac{\sqrt{3}}{2}\right)\right|^2 = 4\pi^2 \left(\cosh(\pi\sqrt{3}) - \cos(\pi\sqrt{5})\right)^{-1}. \end{aligned}$$

Other questions can be conditional in nature. If a cell is known to be a triangle, what is its expected area and its expected perimeter? Brakke [10] computed that these quantities are 0.343089... and 2.740297..., respectively, and subsequent study [15] confirmed these estimates to four decimal places. (The work in [10, 11, 17] has unfortunately remained quite obscure.) See also [35] for more about the distribution of edge lengths L in $\mathbb{R}^d$ and [36] for inradius/circumradius-type analysis of cells in the plane.

The Goudsmit-Miles tessellation of the plane, which is based on the Poisson line process (as opposed to a point process), is discussed in [37].

0.3. Stienen Spheres. Around each particle $p_i \in \mathbb{R}^d$, construct a sphere with diameter equal to the distance to the nearest neighbor p_j of p_i, $i \neq j$. The union of all such spheres and their interiors is called the **Stienen model**. Each sphere is a subset of a Voronoi cell; each cell is a superset of a Stienen sphere. For arbitrary d, if M' denotes the volume of a typical sphere, then $\mathrm{E}(M') = 2^{-d}$ and $\mathrm{Var}(M') = 2^{-2d}$. If $d = 1$, the cross-correlation ρ between M and M' is simply $1/\sqrt{2}$. For $d = 2$ and 3, Olsbo [38] computed $\rho = 0.705143...$ and $\rho = 0.677790...$ via complicated numerical integration. It is not obvious that these correlations are necessarily positive because two neighboring particles lying close together often yield small spheres and large cells.

0.4. Appeal for Help. Any assistance in recovering Brakke's original integrals for $\mathrm{E}(W^2)$, $\mathrm{E}(V^2)$, $\mathrm{E}(Q^2)$, $\mathrm{E}(P^2)$, $\mathrm{E}(B^2)$, $\mathrm{E}(A^2)$ and $\mathrm{E}(M^2)$ when $d = 3$ would be deeply appreciated!

References

[1] M. H. DeGroot, *Probability and Statistics*, Addison-Wesley, 1975, pp. 209–211.

[2] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*, Springer-Verlag, 1988, pp. 18–24; MR0950166 (90e:60060).

[3] P. Hertz, Über den gegenseitigen durchschnittlichen Abstand von Punkten, die mit bekannter mittlerer Dichte im Raume angeordnet sind, *Math. Annalen* 67 (1909) 387–398.

[4] S. Chandrasekhar, Stochastic problems in physics and astronomy, *Rev. Mod. Phys.* 15 (1943) 1–89; also in *Selected Papers on Noise and Stochastic Processes*, ed. N. Wax, Dover, 1954, pp. 3–91.

[5] E. Parzen, *Stochastic Processes*, Holden-Day, 1962, pp. 32–34; MR0139192 (25 #2628).

[6] A. Okabe, B. Boots, K. Sugihara and S. N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd ed., Wiley, 2000, pp. 31–33, 44–50, 291–294, 311–318; MR1210959 (94a:52033) and MR1770006 (2001c:52013).

[7] D. Stoyan, W. S. Kendall and J. Mecke, *Stochastic Geometry and its Applications*, Wiley, 1987, pp. 328–333; MR0895588 (88j:60034a).

[8] J. L. Meijering, Interface area, edge length, and number of vertices in crystal aggregates with random nucleation, *Philips Research Reports* 8 (1953) 270–290.

[9] E. N. Gilbert, Random subdivisions of space into crystals, *Annals Math. Statist.* 33 (1962) 958–972; MR0144253 (26 #1800).

[10] K. A. Brakke, Statistics of random plane Voronoi tessellations, unpublished manuscript (1985), available online at http://www.susqu.edu/brakke/papers/voronoi.htm.

[11] K. A. Brakke, Plane Voronoi tessellation second order statistics, *Abstracts Amer. Math. Soc.* 7 (1986) 235–236.

[12] L. Heinrich, R. Körner, N. Mehlhorn and L. Muche, Numerical and analytical computation of some second-order characteristics of spatial Poisson-Voronoi tessellations, *Statistics* 31 (1998) 235–259; MR1718316 (2000g:60016).

[13] L. Heinrich and L. Muche, Second-order properties of the point process of nodes in a stationary Voronoi tessellation, *Math. Nachr.* 281 (2008) 350–375; erratum 283 (2010) 1674–1676; MR2392118 (2008k:60025); available online at http://www.math.uni-augsburg.de/stochastik/heinrich/publikationen.html.

[14] J. Møller, *Lectures on Random Voronoĭ Tessellations.*, Lect. Notes in Statistics 87, Springer-Verlag, 1994, pp. 83–103; MR1295245 (96b:60028).

[15] A. Hayen and M. P. Quine, Areas of components of a Voronoi polygon in a homogeneous Poisson process in the plane, *Adv. Appl. Probab.* 34 (2002) 281–291; MR1909915 (2003d:60025).

[16] E. Pineda, P. Bruna and D. Crespo, Cell size distribution in random tessellations of space, *Phys. Rev. E* 70 (2004) 066119.

[17] K. A. Brakke, Statistics of three dimensional random Voronoi tessellations, unpublished manuscript (1985), available online at http://www.susqu.edu/brakke/papers/voronoi.htm.

[18] R. E. Miles and R. J. Maillardet, The basic structures of Voronoi and generalized Voronoi polygons, *J. Appl. Probab.* special volume 19A, (1982) 97–111; MR0633183 (83d:60021).

[19] A. Hayen and M. P. Quine, The proportion of triangles in a Poisson-Voronoi tessellation of the plane, *Adv. Appl. Probab.* 32 (2000) 67–74; MR1765171 (2001h:60019).

[20] A. Hayen and M. P. Quine, Calculating the proportion of triangles in a Poisson-Voronoi tessellation of the plane, *J. Statist. Comput. Simul.* 67 (2000) 351–358; MR1815169.

[21] A. L. Hinde and R. E. Miles, Monte-Carlo estimates of the distributions of the random polygons of the Voronoi tessellation with respect to a Poisson process, *J. Statist. Comput. Simul.* 10 (1980) 205–223.

[22] H. G. Hanson, Voronoi cell properties from simulated and real random spheres and points, *J. Stat. Phys.* 30 (1983) 591–605; MR0710751 (85c:52012).

[23] K. A. Brakke, 2,000,000,000 random Voronoi polygons, unpublished manuscript (1985), available online at http://www.susqu.edu/brakke/papers/voronoi.htm.

[24] U. Lorz, Distribution of cell characteristics of the spatial Poisson-Voronoi tessellation and plane sections, *Geometrical Problems of Image Processing*, Proc. 1991 Georgenthal workshop, ed. U. Eckhardt, A. Hübler, W. Nagel and G. Werner, Akademie-Verlag, 1991, pp. 171-178; MR1111686 (91m:68007).

[25] U. Lorz and U. Hahn, Geometric characteristics of spatial Voronoi tessellations and planar sections, Fakultät für Mathematik und Informatik, Technische Universität Bergakademie Freiberg preprint 93-05 (1993), available online at http://www.math.uni-augsburg.de/stochastik/hahn/papers/Lorz_Hahn_1993.pdf.

[26] S. Kumar, S. K. Kurtz, J. R. Banavar and M. G. Sharma, Properties of a three-dimensional Poisson-Voronoi tessellation: a Monte Carlo study, *J. Stat. Phys.* 67 (1992) 523–551.

[27] S. Kumar and S. K. Kurtz, Properties of a two-dimensional Poisson–Voronoi tessellation: a Monte-Carlo study, *Materials Characterization* 31 (1993) 55–68.

[28] S. Kumar and S. K. Kurtz, Monte-Carlo study of angular and edge length distributions in a three-dimensional Poisson-Voronoi tesselation, *Materials Characterization* 34 (1995) 15–27.

[29] M. Tanemura, Statistical distributions of Poisson Voronoi cells in two and three dimensions, *Forma* 18 (2003) 221–247; available online at http://www.scipress.org/journals/forma/pdf/1804/18040221.pdf.

[30] I. Saxl and P. Ponizil, Voronoi tessellations generated by 3D point processes, available online at http://fyzika.ft.utb.cz/voronoi/.

[31] P. Calka, An explicit expression for the distribution of the number of sides of the typical Poisson-Voronoi cell, *Adv. Appl. Probab.* 35 (2003) 863–870; MR2014258 (2004g:60019).

[32] P. Calka, Precise formulae for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process, *Adv. Appl. Probab.* 35 (2003) 551–562; MR1990603 (2004k:60021).

[33] H. J. Hilhorst, The perimeter of large planar Voronoi cells: a double-stranded random walk, *J. Stat. Mech.* (2005) L02003; cond-mat/0412569.

[34] H. J. Hilhorst, New Monte Carlo method for planar Poisson-Voronoi cells, *J. Phys. A* 40 (2007) 2615–2638; cond-mat/0612422; MR2325498 (2008f:60016).

[35] M. Schlather, A formula for the edge length distribution function of the Poisson Voronoi tessellation, *Math. Nachr.* 214 (2000) 113–119; MR1762055 (2001b:60021).

[36] P. Calka, The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane, *Adv. Appl. Probab.* 34 (2002) 702–717; MR1938938 (2003j:60013).

[37] S. R. Finch, Random triangles V, unpublished note (2010).

[38] V. Olsbo, On the correlation between the volumes of the typical Poisson-Voronoi cell and the typical Stienen sphere, *Adv. Appl. Probab.* 39 (2007) 883–892; MR2381579 (2009a:60009).