Dyadic UMD Constants

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For integers $k \geq 1$ and $1 \leq j \leq 2^{k-1}$, define the **Haar functions** $\chi_k^j : [0,1) \to \mathbb{R}$ by

$$\chi_k^j(t) = \begin{cases} 2^{(k-1)/2} & \text{if } \dfrac{j-1}{2^{k-1}} \leq t < \dfrac{j-1/2}{2^{k-1}}, \\ -2^{(k-1)/2} & \dfrac{j-1/2}{2^{k-1}} \leq t < \dfrac{j}{2^{k-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

In words, for each **dyadic subinterval** I of $[0,1)$ of length $2^{-(k-1)}$, we have a function equal to $2^{(k-1)/2}$ on the left half of I and $-2^{(k-1)/2}$ on the right half of I.

Let X and Y be real Banach spaces. A **dyadic martingale** is a set $\{f_n\}_{n=1}^{\infty}$ where each $f_n : [0,1) \to X$ is a linear combination of Haar functions:

$$f_n(t) = \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} \chi_k^j(t) x_k^j$$

and each $x_k^j \in X$ is independent of n. Let $f_0 = 0$ and denote by $d_k = f_k - f_{k-1}$ the martingale differences. Given an operator $T : X \to Y$, the n^{th} **dyadic UMD constant** $\mu_n(T)$ is the least quantity $c \geq 0$ such that

$$\left\| \sum_{k=1}^{n} \varepsilon_k T\, d_k \right\|_2 \leq c \left\| \sum_{k=1}^{n} d_k \right\|_2$$

for all martingale differences $d_1, \ldots, d_n$ and all sequences $\varepsilon_1, \ldots, \varepsilon_n$ of signs. The norm on the right-hand side is the L_2-norm on measurable X-valued functions, the norm on the left-hand side is the L_2-norm on measurable Y-valued functions, and the acronym UMD stands for "unconditional martingale differences".

We are interested in the case when $X = \ell_1^m$ and $Y = \ell_\infty^m$, sequence spaces of m dimensions, and T is the finite summation operator

$$T_m(\xi_1, \ldots, \xi_m) = \left(\xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \xi_3 \ldots, \sum_{i=1}^{m} \xi_i \right)$$

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where $m = 2^n$ for notational convenience. It is known that there exist constants $a > 0$, $b > 0$ such that

$$\sqrt{n} \le a\,\mu_n(T_m), \qquad b\,\mu_n(T_m) \le n$$

independent of n. What, however, is the true asymptotic behavior of $\mu_n(T_m)$?

Wenzel [1, 2] proved that the growth rate of $\mu_n(T_m)$ is the same as the growth rate of

$$\theta_n = \sup_{\pi} \frac{1}{2^n} \sum_{i=0}^{2^n-1} \sup_{0 \le k < 2^n} \left| \sum_{j:\pi(j) \le k} (-2)^{-\kappa(i \oplus j)} \right|$$

where the outer summation is taken over all permutations π of the set $\{0, \ldots, 2^n - 1\}$, $i \oplus j$ denotes the bitwise XOR sum of i and j (addition modulo two without carries [3]), and $\kappa(n) = 1 + \lfloor \ln(n)/\ln(2) \rfloor$ if $n > 0$, $\kappa(0) = 2$. He computed that

$$\theta_3 \approx 0.5937, \qquad \theta_4 \approx 0.6718, \qquad \theta_5 \approx 0.7509, \qquad \theta_6 \approx 0.8203$$

and therefore conjectured that $\sqrt{n}$ is the correct growth rate. In fact, his calculations suggest that $\theta_n \sim (0.3...)\sqrt{n}$ as $n \to \infty$, and we wonder if the corresponding constant for $\mu_n(T_m)/\sqrt{n}$ will ever be known [4].

References

[1] J. Wenzel, The UMD constants of the summation operators, *Quaest. Math.* 27 (2004) 111–136; arXiv:math/0407481; MR2091691 (2005h:46019).

[2] J. Wenzel, *Haar Functions, Martingales and Geometry of Banach Spaces*, Habilitation thesis, Friedrich-Schiller-Universität Jena, 2003; available online at http://users.minet.uni-jena.de/˜wenzel/publications.shtml.

[3] S. R. Finch, Plouffe's constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 430–433.

[4] S. Geiss, S. Montgomery-Smith and E. Saksman, On singular integral and martingale transforms, arXiv:math/0701516.