Constant of Theodorus

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In the complex plane, consider the recursive sequence

$$z_n = \left(1 + \frac{i}{\sqrt{n}}\right) z_{n-1}, \qquad n \geq 1,$$

with starting point $z_0 = 1$. The points z_{n-1} and z_n determine a right triangle relative to the origin 0, with legs 1 and $\sqrt{n}$. Clearly the polar coordinates (r_n, θ_n) of z_n are given by

$$r_n = \sqrt{n+1}, \qquad \theta_n = \begin{cases} \displaystyle\sum_{j=0}^{n-1} \arctan\left(\frac{1}{\sqrt{j+1}}\right) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$$

A closed-form expression for z_n is

$$z_n = \prod_{k=1}^{n} \left(1 + \frac{i}{\sqrt{k}}\right) \qquad n \geq 1,$$

and determines what is called the **discrete spiral of Theodorus**.

Davis [1, 2] and Heuvers, Moak & Boursaw [3] independently constructed the continuous analog of this spiral. A parametric representation is [1, 2]

$$\begin{aligned} f(t) &= \prod_{k=1}^{\infty} \frac{1 + \frac{i}{\sqrt{k}}}{1 + \frac{i}{\sqrt{k+t}}}, \qquad -1 < t < \infty, \\ &= \sqrt{1+t}\, \exp\left(i \sum_{k=1}^{\infty} \left(\arctan\left(\sqrt{k+t}\right) - \arctan\left(\sqrt{k}\right)\right)\right) \end{aligned}$$

and a polar representation is [3]

$$\theta(r) = \sum_{j=0}^{\infty} \left(\arctan\left(\frac{1}{\sqrt{j+1}}\right) - \arctan\left(\frac{1}{\sqrt{j+r^2}}\right)\right), \qquad r > 0.$$

Gronau [2] proved that $f(t)$ is the unique solution of the functional equation

$$f(t) = \left(1 + \frac{i}{\sqrt{t}}\right) f(t-1), \qquad f(0) = 1, \qquad 0 < t < \infty$$

such that $|f(t)|$ is increasing and $\arg(f(t))$ is both increasing and continuous.

Among many possible questions, Davis [1] asked: What is the slope of the spiral at the point 1? Clearly

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = \left.\frac{d\theta}{dr}\right|_{(r,\theta)=(1,0)} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2} + k^{1/2}}$$

which Gautschi [4] evaluated to be 1.8600250792.... This is called the **constant of Theodorus**.

Also, what can be said about the growth of θ_n as $n \to \infty$? For convenience, given a real number ξ, let $\{\xi\} = \xi \bmod 1$ denote the fractional part of ξ. Hlawka [5] proved that

$$\theta_n = 2\sqrt{n+1} + K + \frac{1}{6\sqrt{n+1}} + O\left(n^{-3/2}\right),$$

where the **square root spiral constant** $K = K_0 - 1 - 3\pi/8 = -2.1577829966...$ and

$$K_0 = \frac{1}{8} \int_2^{\infty} \{x\} \left(1 - \{x\}\right) (3x-2) \frac{1}{x^2 (x-1)^{3/2}} dx = 0.0203142484....$$

The numerical estimate of K was obtained by Grünberg [6], correcting an apparent error in [5].

References

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