Slicing Problem

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Before stating the slicing problem, let us examine a related problem with known solution. Let K be a compact convex set in $\mathbb{R}^n$ with nonempty interior. Assume that the n-dimensional volume of K is unity, that is, $\mathrm{vol}_n(K) = 1$. The **centroid** of K is $\mu = \mathrm{E}(X)$, where X is a uniformly distributed random point in K. Let H be any $(n-1)$-dimensional plane passing through μ with corresponding half-spaces H^+ and H^-. Grünbaum [1], Hammer [2] and Mityagin [3] independently proved that

$$\min\left\{\mathrm{vol}_n(K \cap H^+), \mathrm{vol}_n(K \cap H^-)\right\} \geq \left(\frac{n}{n+1}\right)^n \rightarrow \frac{1}{e} = 0.3678794411...$$

and, further, the bound $(n/(n+1))^n$ is best possible. In words, at least a proportion $1/e$ of the convex set volume lies on each side of any planar cut through the centroid. Applications of this result appear in [4, 5, 6, 7, 8, 9]. Grünbaum wrote that it would be interesting to find the analog of this result when substituting $(n-1)$-dimensional surface area for n-dimensional volume, and added that this problem is unsolved even for $n = 2$.

We now give the slicing problem (which is perhaps related to Grünbaum's foreshadowing but likewise unsolved). Let K be as before, with the additional condition that K is **isotropic**:

$$\Sigma = \mathrm{Cov}(X) = \mathrm{E}\left((X-\mu)(X-\mu)^T\right) = \sigma^2 I,$$

where X is a uniformly distributed random point in K and I is the $n \times n$ identity matrix. This latter condition is equivalent to saying that, for every vector $v \in \mathbb{R}^n$,

$$\mathrm{E}\left([v^T(X-\mu)]^2\right) = \sigma^2 |v|^2.$$

The vector μ is often called the **barycenter** of K, the matrix Σ the **inertia matrix** and the scalar σ the **isotropic constant**. Let H be as before. It is conjectured that such an H exists so that

$$\mathrm{vol}_{n-1}(K \cap H) > c$$

for some constant $c > 0$ independent of n and K. (Note that Grünbaum's theorem was true for all H and involved vol_n, not vol_{n-1}.) Bourgain [10, 11] and Paouris [12] proved that

$$\mathrm{vol}_{n-1}(K \cap H) > \frac{b}{n^{1/4}\ln(n)}$$

for some constant $b > 0$. The slicing problem is also known as the hyperplane conjecture; an equivalent formulation is that the isotropic constant $\sigma < a$ for some constant $a < \infty$ independent of n and K.

We mention an obvious converse of Grünbaum's theorem: There exists H for which

$$\mathrm{vol}_n(K \cap H^+) = \mathrm{vol}_n(K \cap H^-) = \frac{1}{2}.$$

A converse of the slicing problem can be expressed as [13, 14]

$$\sigma \geq \frac{1}{\sqrt{n+2}}\omega_n^{-1/n} \to \frac{1}{\sqrt{2\pi e}} = 0.2419707245... = (4.1327313541...)^{-1}$$

where $\omega_n = \pi^{n/2}\Gamma(n/2+1)^{-1}$ is the volume of the unit n-ball. The requirement that K be isotropic is not too restrictive, since every convex set has a linear image which is isotropic. See [15, 16] for applications and [17, 18, 19, 20] for recent progress.

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