

## Optimal Stopping Constants

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Consider the well-known **secretary problem**. An unordered sequence of **applicants** (distinct real numbers)  $s_1, s_2, \dots, s_n$  are interviewed by you one at a time. You have no prior information about the  $s$ s. You know the value of  $n$ , and as  $s_k$  is being interviewed, you must either accept  $s_k$  and end the process, or reject  $s_k$  and interview  $s_{k+1}$ . The decision to accept or reject  $s_k$  must be based solely on whether  $s_k > s_j$  for all  $1 \leq j < k$  (that is, on whether  $s_k$  is a **candidate**). An applicant once rejected cannot later be recalled.

If your objective is to select the most highly qualified applicant (the largest  $s_k$ ), then the optimal strategy is to reject the first  $m - 1$  applicants and accept the next candidate, where [1, 2, 3, 4]

$$m = \min \left\{ k \geq 1 : \sum_{j=k+1}^n \frac{1}{j-1} \leq 1 \right\} \sim \frac{n}{e}$$

as  $n \rightarrow \infty$ . The asymptotic probability of obtaining the best applicant via this strategy is hence  $1/e = 0.3678794411\dots$ , where  $e$  is the natural logarithmic base [1.3]. See a generalization of this in [5, 6, 7].

If your objective is instead to minimize the expected rank  $R_n$  of the chosen applicant (the largest  $s_k$  has rank 1, the second-largest has rank 2, etc.), then different formulation applies. Lindley [8] and Chow et al. [9] derived the optimal strategy in this case and proved that [10]

$$\lim_{n \rightarrow \infty} R_n = \prod_{k=1}^{\infty} \left( 1 + \frac{2}{k} \right)^{\frac{1}{k+1}} = 3.8695192413\dots = C.$$

A variation might include you knowing in advance that  $s_1, s_2, \dots, s_n$  are independent, uniformly distributed variables on the interval  $[0, 1]$ . This is known as a **full-information problem** (as opposed to the no-information problems just discussed). How does knowledge of the distribution improve your chances of success? For the “nothing but the best” objective, Gilbert & Mosteller [11] calculated the asymptotic probability of success to be [12, 13]

$$e^{-a} - (e^a - a - 1) \operatorname{Ei}(-a) = 0.5801642239\dots,$$

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where  $a = 0.8043522628\dots$  is the unique real solution of the equation  $\text{Ei}(a) - \gamma - \ln(a) = 1$ ,  $\text{Ei}$  is the exponential integral [6.2], and  $\gamma$  is the Euler-Mascheroni constant [1.5].

The full-information analog for  $\lim_{n \rightarrow \infty} R_n$  appears to be an open problem [14, 15, 16]. Yet another objective, however, might be to maximize the hiree's expected quality  $Q_n$  itself (the  $k^{\text{th}}$  applicant has quality  $s_k$ ). Clearly

$$Q_0 = 0, \quad Q_n = \frac{1}{2}(1 + Q_{n-1}^2) \quad \text{if } n \geq 1,$$

and  $Q_n \rightarrow 1$  as  $n \rightarrow \infty$ . Moser [11, 17, 18, 19] deduced that

$$Q_n \sim 1 - \frac{2}{n + \ln(n) + b},$$

where the constant  $b$  is estimated [10] to be 1.76799378....

Here is a closely related problem. Assume  $s_1, s_2, \dots, s_n$  are independent, uniformly distributed variables on the interval  $[0, N]$ . Your objective is to minimize the number  $T_N$  of interviews necessary to select an applicant of expected quality  $\geq N - 1$ . Gum [20] sketched a proof that  $T_N = 2N - O(\ln(N))$  as  $N \rightarrow \infty$ . Alternatively, assume everything as before except that  $s'_1, s'_2, \dots, s'_n$  are drawn with replacement from the set  $\{1, 2, \dots, N\}$ . It can be proved here that  $T'_N = cN + O(\sqrt{N})$ , where [10]

$$c = 2 \sum_{k=3}^{\infty} \frac{\ln(k)}{k^2 - 1} - \frac{\ln(2)}{3} = 1.3531302722\dots = \ln(C).$$

The secretary problem and its offshoots fall within the theory of **optimal stopping** [19]. Here is a sample exercise: We observe a fair coin being tossed repeatedly and can stop observing at any time. When we stop, the payoff is the average number of heads observed. What is the best strategy to maximize the expected payoff? Chow & Robbins [21, 22] described a strategy that achieves an expected payoff  $> 0.79 = (0.59 + 1)/2$ .

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