

Random Triangles

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Let $X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3$ be independent normally distributed random variables with mean 0 and variance 1. The points $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)$ constitute the vertices of a triangle in Euclidean 2-space (the plane); the points $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3)$ constitute the vertices of a triangle in Euclidean 3-space. A number of parameters (for example, sides, angles, perimeter and area) describe the triangle, but the corresponding probability density functions are not well-known. We attempt to remedy this situation in this essay. Perhaps the most famous results for random Gaussian triangles are the following [1, 2]:

$$P(\text{a Gaussian triangle in 2-space is obtuse}) = 3/4 = 0.75,$$

$$P(\text{a Gaussian triangle in 3-space is obtuse}) = 1 - 3\sqrt{3}/(4\pi) = 0.5865033284\dots$$

which translate into statements about the maximum angle exceeding $\pi/2$. Consider, however, an arbitrary angle α in a triangle. What is its first moment $E(\alpha)$? This turns out to be trivial. What is its second moment $E(\alpha^2)$? This is more difficult, even in 2 dimensions, and the answer is apparently new. Our essay, the first in a series, arises in an effort to expand upon [3].

0.1. Sides. Let a, b, c denote the sides of a random Gaussian triangle. The trivariate density $f(x, y, z)$ for a, b, c in 2 dimensions is [4]

$$\begin{cases} \frac{2}{3\pi} \frac{xyz}{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}} \exp\left(-\frac{1}{6}(x^2+y^2+z^2)\right) & \text{if } |x-y| < z < x+y, \\ 0 & \text{otherwise} \end{cases}$$

and we shall give an elementary proof of this later. The condition $|x-y| < z < x+y$ is equivalent to $|x-z| < y < x+z$ and to $|y-z| < x < y+z$ via the Law of Cosines. As a consequence, the univariate density for a corresponds to Rayleigh's distribution:

$$\frac{x}{2} \exp\left(-\frac{x^2}{4}\right), \quad x > 0$$

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and

$$\begin{aligned} E(a) &= \sqrt{\pi} = 1.7724538509\dots, & E(a^2) &= 4, \\ E(ab) &= 4E\left(\frac{1}{2}\right) - \frac{3}{2}K\left(\frac{1}{2}\right) = 3.3412233051\dots \end{aligned}$$

where

$$\begin{aligned} K(\xi) &= \int_0^{\pi/2} \frac{1}{\sqrt{1 - \xi^2 \sin^2(\theta)}} d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\xi^2 t^2)}} dt, \\ E(\xi) &= \int_0^{\pi/2} \sqrt{1 - \xi^2 \sin^2(\theta)} d\theta = \int_0^1 \sqrt{\frac{1 - \xi^2 t^2}{1 - t^2}} dt \end{aligned}$$

are complete elliptic integrals of the first and second kind [5]. The cross-correlation coefficient

$$\rho(a, b) = \frac{\text{Cov}(a, b)}{\sqrt{\text{Var}(a) \text{Var}(b)}} = \frac{E(ab) - \pi}{4 - \pi} = 0.2325593465\dots$$

is quite small, indicating weak positive dependency. Interestingly, $\rho(a^2, b^2) = 1/4 = 0.25$ since a^2, b^2 are quadratic forms in normal variables and classical theory applies [6, 7].

The trivariate density for a, b, c in 3 dimensions is [4]

$$\begin{cases} \frac{\sqrt{3}}{9\pi} x y z \exp\left(-\frac{1}{6}(x^2 + y^2 + z^2)\right) & \text{if } |x - y| < z < x + y, \\ 0 & \text{otherwise} \end{cases}$$

which is surprisingly simpler than the corresponding result in 2 dimensions. As a consequence, the univariate density for a corresponds to the Maxwell-Boltzmann distribution:

$$\frac{x^2}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right), \quad x > 0$$

and

$$\begin{aligned} E(a) &= \frac{4}{\sqrt{\pi}} = 2.2567583341\dots, & E(a^2) &= 6, \\ E(ab) &= 2 + \frac{6\sqrt{3}}{\pi} = 5.3079733725\dots, \\ \rho(a, b) &= \frac{-8 + 3\sqrt{3} + \pi}{-8 + 3\pi} = 0.2370510252\dots, & \rho(a^2, b^2) &= \frac{1}{4} = 0.25. \end{aligned}$$

0.2. Perimeter and Area. For perimeter $a + b + c$, the density is a double integral:

$$\int_0^x \int_0^{x-v} f(x-u-v, u, v) du dv, \quad x > 0$$

which we have not attempted to evaluate. Thus only moments are given. In 2 dimensions,

$$E(\text{perimeter}) = 3\sqrt{\pi} = 5.3173615527\dots,$$

$$\begin{aligned} E(\text{perimeter}^2) &= E((a+b+c)^2) \\ &= 3E(a^2) + 6E(ab) \\ &= 12 + 24E\left(\frac{1}{2}\right) - 9K\left(\frac{1}{2}\right) = 32.0473398308\dots \end{aligned}$$

and in 3 dimensions,

$$E(\text{perimeter}) = \frac{12}{\sqrt{\pi}} = 6.7702750025\dots,$$

$$E(\text{perimeter}^2) = 30 + \frac{36\sqrt{3}}{\pi} = 49.8478402351\dots$$

More can be said about area $(1/4)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$. In 2 dimensions, area can be proved to be exponentially distributed, with density [8]

$$\frac{2}{\sqrt{3}} \exp\left(-\frac{2}{\sqrt{3}}x\right), \quad x > 0.$$

The formula given in [9] is unfortunately incorrect. In particular,

$$E(\text{area}) = \frac{\sqrt{3}}{2} = 0.8660254037\dots, \quad E(\text{area}^2) = \frac{3}{2} = 1.5.$$

A proposed density in [10] for 3 dimensional area also seems to be wrong. We find instead

$$E(\text{area}) = \sqrt{3} = 1.7320508075\dots, \quad E(\text{area}^2) = \frac{9}{2} = 4.5$$

and provide experimental verification elsewhere [11].

0.3. Angles. Let α, β, γ denote the angles of a random Gaussian triangle. Of course, $\alpha + \beta + \gamma = \pi$, thus γ can be eliminated from consideration. The bivariate density $\varphi(x, y)$ for α, β in 2 dimensions is [12]

$$\begin{cases} \frac{6}{\pi} \frac{\sin(x) \sin(y) \sin(x+y)}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^2} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x+y < \pi, \\ 0 & \text{otherwise} \end{cases}$$

and we shall confirm this later. The univariate density for α was first discovered by W. S. Kendall [13], via a fairly geometric argument, but has never appeared explicitly in the open literature (the closest was [14]; see also [15]). Starting from the bivariate density, we obtain the univariate density via

$$\begin{aligned} & \frac{6}{\pi} \int_0^{\pi-x} \frac{\sin(x) \sin(y) \sin(x+y)}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^2} dy \\ &= \frac{6}{\pi} \int_0^{\pi-x} \frac{\cos(x) \sin(x)}{2(4-\cos(x)^2)(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)} dy \\ & \quad + \frac{6}{\pi} \int_0^{\pi-x} \left(\frac{\sin(x) \sin(y) \sin(x+y)}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^2} - \frac{\cos(x) \sin(x)}{2(4-\cos(x)^2)(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)} \right) dy \\ &= \frac{3}{\pi} \frac{\cos(x)}{(4-\cos(x)^2)^{3/2}} \left(\frac{\pi}{2} + \arcsin\left(\frac{\cos(x)}{2}\right) \right) + \frac{3}{\pi} \frac{1}{4-\cos(x)^2}. \end{aligned}$$

Call this latter expression $g(x)$. Now, since $3\mathbb{E}(\alpha) = \mathbb{E}(\alpha + \beta + \gamma) = \pi$, we have $\mathbb{E}(\alpha) = \pi/3$. It is harder to show that

$$\mathbb{E}(\alpha^2) = \frac{7}{36}\pi^2 - \frac{1}{2} \text{Li}_2\left(\frac{1}{4}\right) = 1.7852634251\dots$$

where

$$\text{Li}_2(\xi) = \sum_{k=1}^{\infty} \frac{\xi^k}{k^2} = -\int_0^{\xi} \frac{\ln(1-t)}{t} dt$$

is the dilogarithm function [16]. Also, since $3\text{Var}(\alpha) + 6\text{Cov}(\alpha, \beta) = \text{Var}(\alpha + \beta + \gamma) = 0$, we have $\rho(\alpha, \beta) = -1/2$; therefore

$$\mathbb{E}(\alpha\beta) = \frac{5}{72}\pi^2 + \frac{1}{4} \text{Li}_2\left(\frac{1}{4}\right) = 0.7523023542\dots$$

Finally,

$$G(x) = \int_0^x g(\xi) d\xi = \frac{1}{\pi} \frac{\sin(x)}{(4-\cos(x)^2)^{1/2}} \left(\frac{\pi}{2} + \arcsin\left(\frac{\cos(x)}{2}\right) \right) + \frac{1}{\pi} x$$

which implies that $P(\alpha > \pi/2) = 1 - G(\pi/2) = 1/4 = 0.25$, where α is arbitrary. This is equal to $(1/3) P(\max(\alpha, \beta, \gamma) > \pi/2)$ because a triangle can have at most one obtuse angle.

The bivariate density for α, β in 3 dimensions is new, as far as we know:

$$\begin{cases} \frac{24\sqrt{3}}{\pi} \frac{\sin(x)^2 \sin(y)^2 \sin(x+y)^2}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^3} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } 0 < x+y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

The univariate density for α is obtained similarly:

$$\begin{aligned} & \frac{24\sqrt{3}}{\pi} \int_0^{\pi-x} \frac{\sin(x)^2 \sin(y)^2 \sin(x+y)^2}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^3} dy \\ = & \frac{24\sqrt{3}}{\pi} \int_0^{\pi-x} \frac{(2+\cos(x)^2) \sin(x)^2}{4(4-\cos(x)^2)^2 (\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)} dy \\ & + \frac{24\sqrt{3}}{\pi} \int_0^{\pi-x} \left(\frac{\sin(x)^2 \sin(y)^2 \sin(x+y)^2}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^3} - \frac{(2+\cos(x)^2) \sin(x)^2}{4(4-\cos(x)^2)^2 (\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)} \right) dy \\ = & \frac{6\sqrt{3}}{\pi} \frac{(2 + \cos(x)^2) \sin(x)}{(4 - \cos(x)^2)^{5/2}} \left(\frac{\pi}{2} + \arcsin\left(\frac{\cos(x)}{2}\right) \right) + \frac{9\sqrt{3}}{\pi} \frac{\cos(x) \sin(x)}{(4 - \cos(x)^2)^2}. \end{aligned}$$

Call this latter expression $h(x)$. We observe that $h(x) = -\sqrt{3}g'(x)$ and wonder about the meaning of such a connection. As before, $E(\alpha) = \pi/3$. It follows that

$$E(\alpha^2) = \frac{\pi}{3} (\pi - \sqrt{3}) = 1.4760687694\dots,$$

$$E(\alpha\beta) = \frac{\pi}{6} \sqrt{3} = 0.9068996821\dots$$

Finally,

$$P(\alpha > \pi/2) = 1 + \sqrt{3} (g(\pi/2) - g(0)) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.1955011094\dots$$

where α is arbitrary. This again is equal to $(1/3) P(\max(\alpha, \beta, \gamma) > \pi/2)$.

0.4. Order Statistics. We will, for brevity's sake, study only maximum/minimum angles in two dimensions and only maximum/minimum sides in three dimensions. Define $\tilde{g}(x)$ to be

$$\begin{aligned} & \frac{3}{\pi} \frac{\cos(x)}{(4 - \cos(x)^2)^{3/2}} \left(\frac{\pi}{2} - \arcsin\left(\frac{\cos(x)}{2}\right) - 2 \arctan\left(\frac{3 \cos(x)}{\sqrt{4 - \cos(x)^2}}\right) \right) \\ & + \frac{3}{\pi} \frac{1 - 4 \cos(x)^2}{(4 - \cos(x)^2)(1 + 2 \cos(x)^2)} \end{aligned}$$

which is positive for $\pi/3 < x < \pi/2$. Given $\alpha > 0$, $\beta > 0$, $\alpha + \beta < \pi$, the angle α is maximum if $\alpha > \beta$ and $\alpha > \pi - \alpha - \beta$. Hence the density for the maximum angle is

$$\begin{cases} 3 \int_0^x \varphi(x, y) dy & \text{if } \pi/3 < x < \pi/2, \\ 3 \int_{\pi-2x}^{\pi-x} \varphi(x, y) dy & \text{if } \pi/2 < x < \pi \end{cases} = \begin{cases} 3\tilde{g}(x) & \text{if } \pi/3 < x < \pi/2, \\ 3g(x) & \text{if } \pi/2 < x < \pi \end{cases}$$

after breaking up the integral of $\varphi(x, y)$ precisely as outlined earlier. This density again was first discovered by Kendall [13] using a different approach. Incidentally, the identity

$$\arcsin\left(\frac{\cos(x)}{2}\right) = \arctan\left(\frac{\cos(x)}{\sqrt{4 - \cos(x)^2}}\right)$$

might lead to a more natural expression for $\tilde{g}(x)$. The value $3g(\pi) = 3/\pi - 1/\sqrt{3} = 0.3775793893\dots$ is called the shape constant (or first collinearity constant) for planar Gaussian triangles [14, 15].

Define $\psi(x)$ to be

$$\begin{aligned} & \frac{3}{\pi} \frac{\cos(x)}{(4 - \cos(x)^2)^{3/2}} \left(\pi - \arcsin\left(\frac{\sqrt{4 - \cos(x)^2} \sin(x)^2}{2}\right) - 2 \arctan\left(\frac{2 + \cos(x)^2}{\cos(x)\sqrt{4 - \cos(x)^2}}\right) \right) \\ & - \frac{3}{\pi} \frac{1 - 4\cos(x)^2}{(4 - \cos(x)^2)(1 + 2\cos(x)^2)} \end{aligned}$$

which is positive for $0 < x < \pi/3$. The angle α is minimum if $\alpha < \beta$ and $\alpha < \pi - \alpha - \beta$. Hence the density for the minimum angle is

$$3 \int_x^{\pi-2x} \varphi(x, y) dy = 3\psi(x)$$

after similar breakup. This result is evidently new. Moments for these distributions remain open.

Advancing up to three dimensions, the density for the maximum side is [4]

$$\frac{3x}{2\sqrt{\pi}} \left[2\sqrt{\frac{3}{\pi}} \left(e^{-x^2/2} - e^{-x^2/3} \right) + x e^{-x^2/4} \operatorname{erf}\left(\frac{\sqrt{3}x}{6}\right) \right]$$

for $x > 0$, and the density for the minimum side is

$$\frac{3x}{2\sqrt{\pi}} \left[2\sqrt{\frac{3}{\pi}} \left(e^{-x^2/2} - e^{-x^2} \right) + x e^{-x^2/4} \operatorname{erfc}\left(\frac{\sqrt{3}x}{2}\right) \right]$$

where erf , erfc are the error and complementary error functions [17].

0.5. Trivariate Details. Our proof closely follows [18]. Consider sides a, b of a random Gaussian triangle in the plane. Using

$$a^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2, \quad b^2 = (X_3 - X_1)^2 + (Y_3 - Y_1)^2$$

we picture vectors \vec{a}, \vec{b} emanating from (X_1, Y_1) to $(X_2, Y_2), (X_3, Y_3)$, respectively. Define $0 < \theta_a < 2\pi$ to be the angle between vector \vec{a} and the x -axis; define $0 < \theta_b < 2\pi$ likewise. Observe that

$$(u_a, u_b) = \left(\frac{X_2 - X_1}{\sqrt{2}}, \frac{X_3 - X_1}{\sqrt{2}} \right), \quad (v_a, v_b) = \left(\frac{Y_2 - Y_1}{\sqrt{2}}, \frac{Y_3 - Y_1}{\sqrt{2}} \right)$$

are independent random vectors satisfying

$$(u_a, u_b), (v_a, v_b) \sim N \left(\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right) \right).$$

Define $s_a = a^2/4$ and $s_b = b^2/4$. Then

$$u_a = \sqrt{2s_a} \cos(\theta_a), \quad v_a = \sqrt{2s_a} \sin(\theta_a), \quad u_b = \sqrt{2s_b} \cos(\theta_b), \quad v_b = \sqrt{2s_b} \sin(\theta_b)$$

and conversely

$$s_a = \frac{u_a^2 + v_a^2}{2}, \quad s_b = \frac{u_b^2 + v_b^2}{2}, \quad \tan(\theta_a) = \frac{v_a}{u_a}, \quad \tan(\theta_b) = \frac{v_b}{u_b}.$$

The Jacobian matrix of the transformation $(u_a, v_a, u_b, v_b) \mapsto (s_a, s_b, \theta_a, \theta_b)$ is

$$J = \begin{pmatrix} u_a & v_a & 0 & 0 \\ 0 & 0 & u_b & v_b \\ -\frac{v_a}{u_a^2 + v_a^2} & \frac{u_a}{u_a^2 + v_a^2} & 0 & 0 \\ 0 & 0 & -\frac{v_b}{u_b^2 + v_b^2} & \frac{u_b}{u_b^2 + v_b^2} \end{pmatrix}$$

For example,

$$\sec(\theta_a)^2 \frac{\partial \theta_a}{\partial u_a} = \frac{\partial}{\partial u_a} \tan(\theta_a) = \frac{\partial}{\partial u_a} \frac{v_a}{u_a} = -\frac{v_a}{u_a^2}$$

implies that

$$\frac{\partial \theta_a}{\partial u_a} = -\cos(\theta_a)^2 \frac{v_a}{u_a^2} = -\frac{u_a^2}{2s_a} \frac{v_a}{u_a^2} = -\frac{v_a}{u_a^2 + v_a^2}.$$

As another example,

$$\sec(\theta_a)^2 \frac{\partial \theta_a}{\partial v_a} = \frac{\partial}{\partial v_a} \tan(\theta_a) = \frac{\partial}{\partial v_a} \frac{v_a}{u_a} = \frac{1}{u_a}$$

implies that

$$\frac{\partial \theta_a}{\partial v_a} = \cos(\theta_a)^2 \frac{1}{u_a} = \frac{u_a^2}{2s_a} \frac{1}{u_a} = \frac{u_a}{u_a^2 + v_a^2}.$$

Since the absolute determinant $|J| = 1$, changing variables from (u_a, v_a, u_b, v_b) to $(s_a, s_b, \theta_a, \theta_b)$ is easily performed. The density of (u_a, u_b) gives rise to

$$\begin{aligned} & \frac{1}{2\pi\sqrt{1 - (\frac{1}{2})^2}} \exp \left[-\frac{1}{2 \left(1 - (\frac{1}{2})^2\right)} \left(u_a^2 - 2(\frac{1}{2})u_a u_b + u_b^2\right) \right] \\ &= \frac{1}{\sqrt{3}\pi} \exp \left[-\frac{2}{3} \left(u_a^2 - u_a u_b + u_b^2\right) \right] \\ &= \frac{1}{\sqrt{3}\pi} \exp \left[-\frac{2}{3} \left(2s_a \cos(\theta_a)^2 - \sqrt{2s_a}\sqrt{2s_b} \cos(\theta_a) \cos(\theta_b) + 2s_b \cos(\theta_b)^2\right) \right] \\ &= \frac{1}{\sqrt{3}\pi} \exp \left[-\frac{4}{3} \left(s_a \cos(\theta_a)^2 - \sqrt{s_a s_b} \cos(\theta_a) \cos(\theta_b) + s_b \cos(\theta_b)^2\right) \right] \end{aligned}$$

and the density of (v_a, v_b) likewise gives rise to

$$\begin{aligned} & \frac{1}{\sqrt{3}\pi} \exp \left[-\frac{2}{3} \left(v_a^2 - v_a v_b + v_b^2\right) \right] \\ &= \frac{1}{\sqrt{3}\pi} \exp \left[-\frac{4}{3} \left(s_a \sin(\theta_a)^2 - \sqrt{s_a s_b} \sin(\theta_a) \sin(\theta_b) + s_b \sin(\theta_b)^2\right) \right]. \end{aligned}$$

By independence, the density of (u_a, u_b, v_a, v_b) is

$$\frac{1}{3\pi^2} \exp \left[-\frac{4}{3} \left(s_a - \sqrt{s_a s_b} \cos(\theta_a - \theta_b) + s_b\right) \right]$$

where $0 < \theta_a < 2\pi$, $0 < \theta_b < 2\pi$.

We move toward integrating out θ_a . Let $\omega = \theta_a - \theta_b$. The Jacobian matrix of the transformation $(s_a, s_b, \theta_a, \theta_b) \mapsto (s_a, s_b, \omega, \theta_a)$ is

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and $|K| = 1$, hence the density of $(s_a, s_b, \omega, \theta_a)$ is

$$\frac{1}{3\pi^2} \exp \left[-\frac{4}{3} \left(s_a - \sqrt{s_a s_b} \cos(\omega) + s_b\right) \right]$$

where $-2\pi < \omega < 2\pi$ plus an additional condition. If $\omega < 0$, then $\theta_b < 2\pi$ forces $\theta_a < 2\pi + \theta_a - \theta_b = 2\pi + \omega$, thus

$$\begin{aligned} & \frac{1}{3\pi^2} \int_0^{2\pi+\omega} \exp\left[-\frac{4}{3}(s_a - \sqrt{s_a s_b} \cos(\omega) + s_b)\right] d\theta_a \\ &= \frac{2\pi + \omega}{3\pi^2} \exp\left[-\frac{4}{3}(s_a - \sqrt{s_a s_b} \cos(\omega) + s_b)\right]; \end{aligned}$$

if $\omega > 0$, then $\theta_b > 0$ forces $\theta_a > \theta_a - \theta_b = \omega$, thus

$$\begin{aligned} & \frac{1}{3\pi^2} \int_\omega^{2\pi} \exp\left[-\frac{4}{3}(s_a - \sqrt{s_a s_b} \cos(\omega) + s_b)\right] d\theta_a \\ &= \frac{2\pi - \omega}{3\pi^2} \exp\left[-\frac{4}{3}(s_a - \sqrt{s_a s_b} \cos(\omega) + s_b)\right]. \end{aligned}$$

In either case, the coefficient numerator is $2\pi - |\omega|$ and the density is symmetric in ω about 0. Let $\gamma = |\omega|$, then we multiply by 2 to obtain the density of (s_a, s_b, γ) :

$$\frac{2(2\pi - \gamma)}{3\pi^2} \exp\left[-\frac{4}{3}(s_a - \sqrt{s_a s_b} \cos(\gamma) + s_b)\right]$$

where $0 < \gamma < 2\pi$. Adding contributions at γ and $2\pi - \gamma$ yields

$$\frac{4}{3\pi} \exp\left[-\frac{4}{3}(s_a - \sqrt{s_a s_b} \cos(\gamma) + s_b)\right]$$

for $0 < \gamma < \pi$, which works since $2(2\pi - \gamma) + 2\gamma = 4\pi$ and $\cos(\gamma) = \cos(2\pi - \gamma)$. Replacing s_a, s_b by $a^2/4, b^2/4$ yields

$$\begin{aligned} & \frac{4}{3\pi} \exp\left[-\frac{4}{3}\left(\frac{a^2}{4} - \frac{ab}{4} \cos(\gamma) + \frac{b^2}{4}\right)\right] \frac{ab}{2 \cdot 2} \\ &= \frac{1}{3\pi} ab \exp\left[-\frac{1}{3}(a^2 - ab \cos(\gamma) + b^2)\right]. \end{aligned}$$

This is already useful for computing moments of area:

$$\mathbb{E}\left(\left(\frac{1}{2}ab \sin(\gamma)\right)^m\right) = m! \left(\frac{\sqrt{3}}{2}\right)^m$$

for all positive integers m . Also, an initial step in calculating $\mathbb{E}(ab)$ is to evaluate

$$\frac{1}{3\pi} \int_0^\pi a^2 b^2 \exp\left[-\frac{1}{3}(a^2 - ab \cos(\gamma) + b^2)\right] d\gamma = \frac{a^2 b^2}{3} \exp\left[-\frac{1}{3}(a^2 + b^2)\right] I_0\left(\frac{ab}{3}\right)$$

where $I_0(z)$ is the modified Bessel function of the first kind [19]. Note that the angle γ is adjacent to sides a, b and opposite to side c , as is traditional. It would be good someday to see the analogous density for (α, β, c) .

We now bring c into the trivariate density, removing γ . Differentiating the Law of Cosines

$$c^2 = a^2 - 2ab \cos(\gamma) + b^2$$

with respect to γ , it is clear that

$$\begin{aligned} 2c \, dc &= 2ab \sin(\gamma) \, d\gamma \\ &= \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \, d\gamma \end{aligned}$$

by a formula for area, and hence the density becomes

$$\begin{aligned} &\frac{1}{3\pi} ab \exp \left[-\frac{1}{3} (a^2 - ab \cos(\gamma) + b^2) \right] \\ &= \frac{1}{3\pi} ab \exp \left[-\frac{1}{6} (a^2 + b^2 + (a^2 - 2ab \cos(\gamma) + b^2)) \right] \\ &= \frac{2}{3\pi} \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \exp \left[-\frac{1}{6} (a^2 + b^2 + c^2) \right] \end{aligned}$$

assuming $0 < \gamma < \pi$, that is, $a^2 - 2ab + b^2 < c^2 < a^2 + 2ab + b^2$. The required condition $|a-b| < c < a+b$ does not change upon permutation of sides a, b, c .

Note that the variables s_a, s_b are each exponentially distributed with mean 1, with cross-correlation $1/4$. A closed-form expression for the density of (s_a, s_b) is not possible [18], but an infinite series representation [20]

$$\sum_{n=0}^{\infty} \frac{1}{4^n} \Phi(-n, 1, s_a) \Phi(-n, 1, s_b) \exp(-(s_a + s_b))$$

is valid, where $\Phi(u, v, w)$ is the confluent hypergeometric function of the first kind [19]. In this special case,

$$\Phi(-n, 1, t) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k!} t^k.$$

Proving the series representation makes use of

$$s_a = \left(\frac{u_a}{\sqrt{2}} \right)^2 + \left(\frac{v_a}{\sqrt{2}} \right)^2, \quad s_b = \left(\frac{u_b}{\sqrt{2}} \right)^2 + \left(\frac{v_b}{\sqrt{2}} \right)^2$$

and the fact that $u_a/\sqrt{2}, u_b/\sqrt{2}$ are jointly normal with mean 0, variance $1/2$ and cross-correlation $1/2$. Other multivariate generalizations of the exponential distribution are found in [22].

For the (a, b, c) -density of random Gaussian triangles in 3-space, we refer to [4].

0.6. Bivariate Details. Let $\Delta = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)$ for convenience. The transformation $(a, b, c) \mapsto (\alpha, \beta, c)$ is prescribed via

$$\cos(\alpha) = \frac{-a^2 + b^2 + c^2}{2bc}, \quad \cos(\beta) = \frac{-b^2 + a^2 + c^2}{2ac}.$$

We have, for example,

$$-\sin(\alpha) \frac{\partial \alpha}{\partial a} = -\frac{a}{bc}, \quad -\sin(\alpha) \frac{\partial \alpha}{\partial b} = \frac{a^2 + b^2 - c^2}{2b^2c}, \quad -\sin(\alpha) \frac{\partial \alpha}{\partial c} = \frac{a^2 - b^2 + c^2}{2bc^2}$$

hence

$$\begin{aligned} \frac{\partial \alpha}{\partial a} &= \frac{a}{bc} \frac{1}{\sin(\alpha)} = \frac{a}{bc} \frac{1}{\sqrt{1 - \cos(\alpha)^2}} = \frac{a}{bc} \frac{2bc}{\sqrt{\Delta}} = \frac{2a}{\sqrt{\Delta}}, \\ \frac{\partial \alpha}{\partial b} &= -\frac{a^2 + b^2 - c^2}{2b^2c} \frac{1}{\sin(\alpha)} = -\frac{a^2 + b^2 - c^2}{2b^2c} \frac{2bc}{\sqrt{\Delta}} = -\frac{a^2 + b^2 - c^2}{b\sqrt{\Delta}}, \\ \frac{\partial \alpha}{\partial c} &= -\frac{a^2 - b^2 + c^2}{2bc^2} \frac{1}{\sin(\alpha)} = -\frac{a^2 - b^2 + c^2}{2bc^2} \frac{2bc}{\sqrt{\Delta}} = -\frac{a^2 - b^2 + c^2}{c\sqrt{\Delta}}. \end{aligned}$$

The corresponding Jacobian matrix is

$$L = \begin{pmatrix} \frac{2a}{\sqrt{\Delta}} & \frac{-a^2 - b^2 + c^2}{b\sqrt{\Delta}} & \frac{-a^2 + b^2 - c^2}{c\sqrt{\Delta}} \\ \frac{-a^2 - b^2 + c^2}{a\sqrt{\Delta}} & \frac{2b}{\sqrt{\Delta}} & \frac{a^2 - b^2 - c^2}{c\sqrt{\Delta}} \\ 0 & 0 & 1 \end{pmatrix}$$

and $|L| = 1/(ab)$. By the Law of Sines,

$$a = c \frac{\sin(\alpha)}{\sin(\gamma)} = c \frac{\sin(\alpha)}{\sin(\alpha + \beta)}, \quad b = c \frac{\sin(\beta)}{\sin(\gamma)} = c \frac{\sin(\beta)}{\sin(\alpha + \beta)}$$

and, under the change of variables,

$$\sqrt{\Delta} = 2c^2 \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)}.$$

The density of (α, β, c) in two dimensions is

$$\begin{aligned} & \frac{2}{3\pi} \frac{a^2 b^2 c}{\sqrt{\Delta}} \exp \left[-\frac{1}{6} (a^2 + b^2 + c^2) \right] \\ &= \frac{2c^5}{3\pi} \frac{\sin(\alpha)^2 \sin(\beta)^2}{\sin(\alpha + \beta)^4 \sqrt{\Delta}} \exp \left[-\frac{c^2}{6 \sin(\alpha + \beta)^2} (\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2) \right] \\ &= \frac{c^3}{3\pi} \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3} \exp \left[-\frac{c^2}{6} \frac{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2}{\sin(\alpha + \beta)^2} \right]. \end{aligned}$$

Integrating out c is facilitated by observing that

$$\int_0^{\infty} c^3 \exp\left(-\frac{c^2}{6}r\right) dc = \frac{18}{r^2}$$

for $r > 0$, therefore the density of (α, β) in two dimensions is

$$\begin{aligned} & \frac{18 \sin(\alpha) \sin(\beta)}{3\pi \sin(\alpha + \beta)^3} \left(\frac{\sin(\alpha + \beta)^2}{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2} \right)^2 \\ &= \frac{6 \sin(\alpha) \sin(\beta) \sin(\alpha + \beta)}{\pi (\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2)^2}. \end{aligned}$$

Similarly, the density of (α, β, c) in three dimensions is

$$\begin{aligned} & \frac{\sqrt{3}}{9\pi} a^2 b^2 c \exp\left(-\frac{1}{6}(a^2 + b^2 + c^2)\right) \\ &= \frac{\sqrt{3}c^5 \sin(\alpha)^2 \sin(\beta)^2}{9\pi \sin(\alpha + \beta)^4} \exp\left[-\frac{c^2 \sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2}{6 \sin(\alpha + \beta)^2}\right]. \end{aligned}$$

Here we observe that

$$\int_0^{\infty} c^5 \exp\left(-\frac{c^2}{6}r\right) dc = \frac{216}{r^3}$$

for $r > 0$, therefore the density of (α, β) in three dimensions is

$$\begin{aligned} & \frac{216\sqrt{3} \sin(\alpha)^2 \sin(\beta)^2}{9\pi \sin(\alpha + \beta)^4} \left(\frac{\sin(\alpha + \beta)^2}{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2} \right)^3 \\ &= \frac{24\sqrt{3}}{\pi} \frac{\sin(\alpha)^2 \sin(\beta)^2 \sin(\alpha + \beta)^2}{(\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2)^3}. \end{aligned}$$

We turn attention to the most interesting of our moment evaluations, that concerning $E(\alpha^2)$. First,

$$\int_0^{\pi} \arcsin\left(\frac{\cos(x)}{2}\right) dx = 0$$

because $\arcsin(\cos(\pi - x)/2) = \arcsin(-\cos(x)/2) = -\arcsin(\cos(x)/2)$. Consequently

$$\begin{aligned} \int_0^{\pi} \frac{x \sin(x)}{\sqrt{4 - \cos(x)^2}} dx &= -x \arcsin\left(\frac{\cos(x)}{2}\right) \Big|_0^{\pi} + \int_0^{\pi} \arcsin\left(\frac{\cos(x)}{2}\right) dx \\ &= \frac{\pi^2}{6} \end{aligned}$$

using integration by parts. Second,

$$\begin{aligned} \int_0^\pi \left(\arcsin \left(\frac{\cos(x)}{2} \right) \right)^2 dx &= \frac{1}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{16^{m+n}} \binom{2m}{m} \binom{2n}{n} \frac{1}{2m+1} \frac{1}{2n+1} \int_0^\pi \cos(x)^{2m+2n+2} dx \\ &= \frac{\pi}{16} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{64^{m+n}} \binom{2m}{m} \binom{2n}{n} \binom{2m+2n+2}{m+n+1} \frac{1}{2m+1} \frac{1}{2n+1} \\ &= \frac{\pi}{2} \operatorname{Li}_2 \left(\frac{1}{4} \right) \end{aligned}$$

which is a curious generalization of sums found in [23]. Consequently

$$\begin{aligned} \int_0^\pi \frac{x \sin(x)}{\sqrt{4 - \cos(x)^2}} \arcsin \left(\frac{\cos(x)}{2} \right) dx &= -\frac{x}{2} \left(\arcsin \left(\frac{\cos(x)}{2} \right) \right)^2 \Big|_0^\pi + \frac{1}{2} \int_0^\pi \left(\arcsin \left(\frac{\cos(x)}{2} \right) \right)^2 dx \\ &= -\frac{\pi^3}{72} + \frac{\pi}{4} \operatorname{Li}_2 \left(\frac{1}{4} \right) \end{aligned}$$

using integration by parts again. Third, $G(\pi) = 1$ and $G(0) = 0$, where $G'(x) = g(x)$. Finally,

$$\begin{aligned} \int_0^\pi x^2 G'(x) dx &= x^2 G(x) \Big|_0^\pi - 2 \int_0^\pi x G(x) dx \\ &= \pi^2 - \frac{2}{\pi} \int_0^\pi x \frac{\sin(x)}{\sqrt{4 - \cos(x)^2}} \left(\frac{\pi}{2} + \arcsin \left(\frac{\cos(x)}{2} \right) \right) dx - \frac{2}{\pi} \int_0^\pi x^2 dx \\ &= \pi^2 - \frac{\pi^2}{6} - 2 \left(-\frac{\pi^2}{72} + \frac{1}{4} \operatorname{Li}_2 \left(\frac{1}{4} \right) \right) - \frac{2}{3} \pi^2 \\ &= \frac{7}{36} \pi^2 - \frac{1}{2} \operatorname{Li}_2 \left(\frac{1}{4} \right) \end{aligned}$$

as was to be shown.

We conclude with an unsolved problem: what is an exact expression for

$$E(a\gamma) = \frac{1}{3\pi} \int_0^\infty \int_0^\infty \int_0^\pi x^2 y \theta \exp \left[-\frac{1}{3} (x^2 - xy \cos(\theta) + y^2) \right] d\theta dy dx = 1.6377\dots$$

(in two dimensions)? An answer for $E(a\alpha)$ is believed to be even more difficult.

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