Chebyshev's Bias

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April 26, 2006

How do we quantify irregularities in the distribution of prime numbers? Define

$$\pi_{q,a}(n) = \#\left\{p \le n : p \equiv a \bmod q\right\}$$

where $\gcd(a, q) = 1$. A well-known result:

$$\lim_{n\to\infty} \frac{\ln(n)}{n}\pi_{q,a}(n) = \frac{1}{\varphi(q)}$$

informs us that primes are asymptotically equidistributed modulo q, where $\varphi(q)$ is the Euler totient. There is, however, unrest beneath the surface of such symmetry. For fixed a_1, a_1, ..., a_r and q, define

$$S_N = \#\left\{n \le N : \pi_{q,a_1}(n) > \pi_{q,a_2}(n) > \ldots > \pi_{q,a_r}(n)\right\}$$

and

$$P(a_1 > a_2 > \ldots > a_r \bmod q) = \lim_{N\to\infty} \frac{1}{\ln(N)} \sum_{n\in S_N} \frac{1}{n}.$$

As the notation suggests, P is to be interpreted as a probability (via logarithmic measure). Rubinstein & Sarnak [1], assuming both the Generalized Riemann Hypothesis and the Grand Simplicity Hypothesis [2], succeeded in proving that

$$P(3 > 1 \bmod 4) = 0.9959280...,$$

$$P(2 > 1 \bmod 3) = 0.9990633....$$

Feuerverger & Martin [3] further proved that

$$P(3 > 5 > 7 \bmod 8) = P(7 > 5 > 3 \bmod 8) = 0.1928013...,$$

$$P(3 > 7 > 5 \bmod 8) = P(5 > 7 > 3 \bmod 8) = 0.1664263...,$$

$$P(5 > 3 > 7 \bmod 8) = P(7 > 3 > 5 \bmod 8) = 0.1407724...$$

and

$$P(5 > 7 > 11 \bmod 12) = P(11 > 7 > 5 \bmod 12) = 0.1984521...,$$

$$\begin{aligned}P(7>5>11 \bmod 12)=P(11>5>7 \bmod 12)=0.1799849...,\\ P(5>11>7 \bmod 12)=P(7>11>5 \bmod 12)=0.1215630...;\end{aligned}$$

thus it is more probable that 5 will occupy the middle position for mod 8, and 7 will occupy the middle position for mod 12!

New constants do not always emerge: we have, for example,

$$P(1>4 \bmod 5)=P(2>3 \bmod 5)=\frac{1}{2}$$

which is due to 1, 4 being squares mod 5 and 2, 3 being nonsquares mod 5. Also

$$P(1>2>4 \bmod 7)=P(3>5>6 \bmod 7)=\frac{1}{6}$$

which is due to 1, 2, 4 being squares mod 7 and 3, 5, 6 being nonsquares mod 7. Examples with exact probabilities $1/r!$, where $r \geq 3$, have not been found.

Define the logarithmic integral

$$\operatorname{li}(x)=\int\limits_2^x \frac{1}{\ln(t)}dt$$

for $x \geq 2$ and

$$T_N=\#\left\{n \leq N : \pi_{1,0}(n)>\operatorname{li}(n)\right\}.$$

In another demonstration of their methods, Rubinstein & Sarnak [1] showed that

$$\lim_{N\to\infty}\frac{1}{\ln(N)}\sum_{n\in T_N}\frac{1}{n}=0.00000026...=1-0.99999973....$$

Further results have been obtained by Ng [4], as reported in [5]; we shall discuss these at a later time.

Addendum Let us return to the usual sense of probability (via uniform measure). Brent [6] conjectured that, for random $0<N<n$, we have

$$\lim_{n\to\infty}\mathrm{P}\left(\frac{\operatorname{li}(N)-\pi_{1,0}(N)}{\sqrt{N}/\ln(N)}<x\right)=F(x)$$

where the probability distribution F has mean $\mu=1$ and variance $\sigma^2\approx(0.21)^2$. If the Riemann hypothesis is true, then it can be shown that [7]

$$\begin{aligned}\sigma^2 &= 2-\ln(4\pi)+\gamma=(0.2149218879...)^2\\ &= 0.0461914179...=2(0.0230957089...)\end{aligned}$$

which we have seen elsewhere [8, 9]. An open question is whether F is the normal distribution; a density plot [1] and a time series graph [5] suggest that the answer might be yes. We also wonder about extensions of this probabilistic result to $\pi_{q,a}(n)$ for arbitrary a and q.

References

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