

Planar Graph Growth Constants

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A **graph** of order n consists of a set of n **vertices** (points) together with a set of **edges** (unordered pairs of distinct points). Note that loops and multiple parallel edges are automatically disallowed. Two vertices joined by an edge are called **adjacent**. Two graphs X and Y are **isomorphic** if there is a one-to-one map from the vertices of X to the vertices of Y that preserves adjacency (see Figure 1). Diagrams for all non-isomorphic graphs of order ≤ 7 appear in [1].

A graph is **connected** if, for any two distinct vertices u and w , there is a sequence of adjacent vertices v_0, v_1, \dots, v_m such that $v_0 = u$ and $v_m = w$ (see Figure 2). The generating function for graphs [2]

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} g_n x^n \\ &= x + 2x^2 + 4x^3 + 11x^4 + 34x^5 + 156x^6 + 1044x^7 + 12346x^8 + 274668x^9 + \dots, \end{aligned}$$

and the generating function for connected graphs

$$\begin{aligned} c(x) &= \sum_{n=1}^{\infty} c_n x^n \\ &= x + x^2 + 2x^3 + 6x^4 + 21x^5 + 112x^6 + 853x^7 + 11117x^8 + 261080x^9 + \dots \end{aligned}$$

are related via the Euler transform [3]

$$1 + g(x) = \exp\left(\sum_{k=1}^{\infty} \frac{c(x^k)}{k}\right).$$

If we agree that $g_0 = 1$, then the coefficients satisfy

$$g_n = \frac{1}{n} \sum_{k=1}^n \left(\sum_{d|k} d c_d \right) g_{n-k}, \quad n \geq 1.$$

Asymptotically, $g_n \sim 2^{n(n-1)/2}/n!$ as $n \rightarrow \infty$, or more precisely [4, 5, 6],

$$g_n \sim \frac{2^{n(n-1)/2}}{n!} \left(1 + 2 \frac{n(n-1)}{2^n} + \frac{8}{3} \frac{n(n-1)(n-2)(3n-7)}{2^{2n}} + O\left(\frac{n^5}{2^{5n/2}}\right) \right).$$

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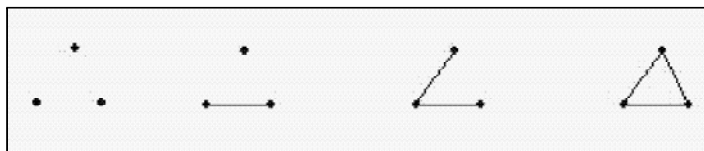


Figure 1: There exist 4 non-isomorphic graphs of order 3, that is, $g_3 = 4$.

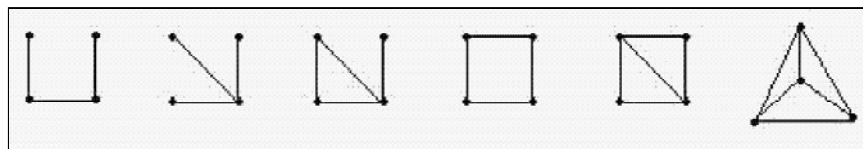


Figure 2: There exist 6 non-isomorphic connected graphs of order 4, that is, $c_4 = 6$.

A **separating set** or **vertex cut** of a graph X is a subset of the vertices of X , the removal of which disconnects X . Let k be a nonnegative integer. A graph is **k -connected** if every vertex cut has at least k vertices. Clearly any graph is 0-connected and 1-connectedness is equivalent to connectedness. A 2-connected graph is often called **biconnected** or **nonseparable** and a 3-connected graph is often called **triconnected**. Observe that, when we count graphs, we do so abstractly; we are not counting embeddings in the plane or on the sphere (Figures 3 and 4).

If we **label** the vertices of a graph distinctly with the integers $1, 2, \dots, n$, the corresponding enumeration problems often simplify; for example, there are exactly $2^{n(n-1)/2}$ labeled graphs. The generating function for labeled graphs

$$G(x) = \sum_{n=1}^{\infty} \frac{G_n}{n!} x^n = \sum_{n=1}^{\infty} \frac{2^{n(n-1)/2}}{n!} x^n$$

and the generating function for connected labeled graphs [7]

$$\begin{aligned} C(x) &= \sum_{n=1}^{\infty} \frac{C_n}{n!} x^n \\ &= x + \frac{1}{2!} x^2 + \frac{4}{3!} x^3 + \frac{38}{4!} x^4 + \frac{728}{5!} x^5 + \frac{26704}{6!} x^6 + \frac{1866256}{7!} x^7 + \dots \end{aligned}$$

satisfy [3, 8]

$$1 + G(x) = \exp(C(x)), \quad G_n = \frac{1}{n} \sum_{k=1}^n \binom{n}{k} k G_{n-k} C_k,$$

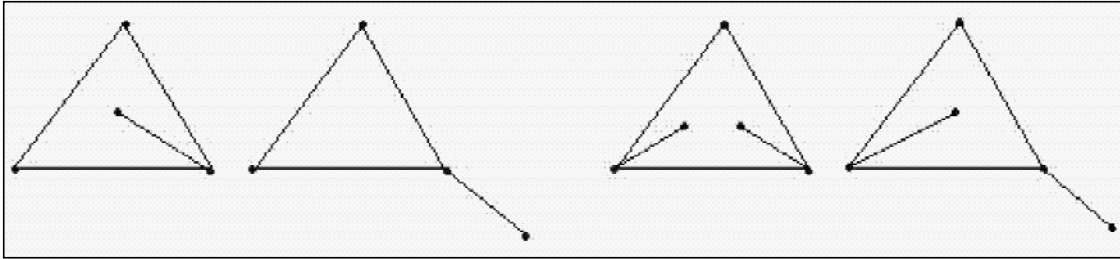


Figure 3: The left-hand pair of 1-connected graphs are isomorphic yet are distinct planar embeddings. The right-hand pair of 1-connected graphs are isomorphic yet are distinct spherical embeddings.

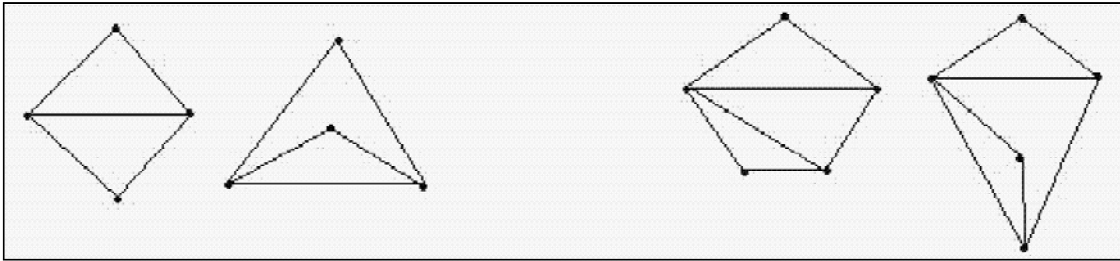


Figure 4: The left-hand pair of 2-connected graphs are isomorphic yet are distinct planar embeddings. The right-hand pair of 2-connected graphs are isomorphic yet are distinct spherical embeddings.

where again we agree that $G_0 = 1$. In fact, $C_n \sim G_n$ as $n \rightarrow \infty$; consequently, almost all graphs are connected [6]. Likewise, almost all graphs are 2-connected.

A graph is **planar** if it can be embedded in the plane (as opposed to a **map**, which is a graph together with its embedding). In other words, a planar graph can be drawn so that no two edges meet except at a vertex at which both are incident. The first example of a nonplanar graph is the complete graph K_5 with 5 vertices and all 10 edges; a second well-known example is the complete bipartite graph $K_{3,3}$ with 6 vertices (three houses and three utilities) and 9 edges (each house is adjacent to each utility). The generating function for planar graphs [9]

$$\begin{aligned} \bar{g}(x) &= \sum_{n=1}^{\infty} \bar{g}_n x^n \\ &= x + 2x^2 + 4x^3 + 11x^4 + 33x^5 + 142x^6 + 822x^7 + 6966x^8 + 79853x^9 + \dots, \end{aligned}$$

the generating function for connected planar graphs

$$\begin{aligned}\bar{c}(x) &= \sum_{n=1}^{\infty} \bar{c}_n x^n \\ &= x + x^2 + 2x^3 + 6x^4 + 20x^5 + 99x^6 + 646x^7 + 5974x^8 + 71885x^9 + \dots,\end{aligned}$$

the generating function for 2-connected planar graphs (see Figure 5)

$$\begin{aligned}\bar{b}(x) &= \sum_{n=1}^{\infty} \bar{b}_n x^n \\ &= x^3 + 3x^4 + 9x^5 + 44x^6 + 294x^7 + 2893x^8 + 36496x^9 + 545808x^{10} + \dots,\end{aligned}$$

and the generating function for 3-connected planar graphs (also called **polyhedra**)

$$\begin{aligned}\bar{a}(x) &= \sum_{n=1}^{\infty} \bar{a}_n x^n \\ &= x^4 + 2x^5 + 7x^6 + 34x^7 + 257x^8 + 2606x^9 + 32300x^{10} + \dots,\end{aligned}$$

do not appear to be easily related. The growth rate of $\{\bar{g}_n\}_{n=1}^{\infty}$, defined as $\gamma_u = \lim_{n \rightarrow \infty} \bar{g}_n^{1/n}$, can be proved to exist and satisfies $\gamma_u \leq 30.0606 = 2^{4.9098}$ [10, 11, 12]. We will discuss lower bounds on this constant shortly. Also, the asymptotics of $\{\bar{a}_n\}_{n=1}^{\infty}$ are precisely known [13, 14, 15]:

$$\bar{a}_n \sim \kappa n^{-7/2} \alpha^n$$

where

$$\alpha = \frac{16}{27} (17 + 7\sqrt{7}) = 21.0490424755\dots = (0.0475080992\dots)^{-1}$$

and κ is a constant (omitted).

The generating function for labeled planar graphs [16]

$$\begin{aligned}\bar{G}(x) &= \sum_{n=1}^{\infty} \frac{\bar{G}_n}{n!} x^n \\ &= x + \frac{2}{2!}x^2 + \frac{8}{3!}x^3 + \frac{64}{4!}x^4 + \frac{1023}{5!}x^5 + \frac{32071}{6!}x^6 + \frac{1823707}{7!}x^7 + \dots,\end{aligned}$$

the generating function for labeled connected planar graphs

$$\begin{aligned}\bar{C}(x) &= \sum_{n=1}^{\infty} \frac{\bar{C}_n}{n!} x^n \\ &= x + \frac{1}{2!}x^2 + \frac{4}{3!}x^3 + \frac{38}{4!}x^4 + \frac{727}{5!}x^5 + \frac{26013}{6!}x^6 + \frac{1597690}{7!}x^7 + \dots,\end{aligned}$$

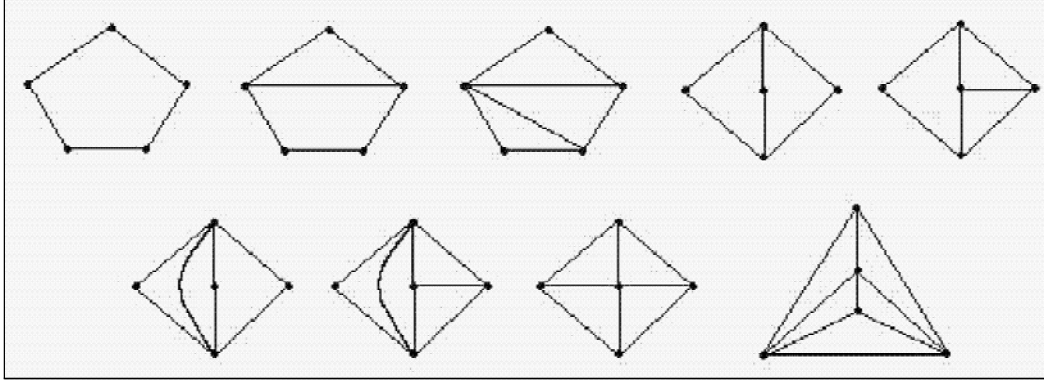


Figure 5: There exist 9 non-isomorphic 2-connected planar graphs of order 5.

and the generating function for labeled 2-connected planar graphs

$$\begin{aligned}\bar{B}(x) &= \sum_{n=1}^{\infty} \frac{\bar{B}_n}{n!} x^n \\ &= \frac{1}{3!} x^3 + \frac{10}{4!} x^4 + \frac{237}{5!} x^5 + \frac{10707}{6!} x^6 + \frac{774924}{7!} x^7 + \frac{78702536}{8!} x^8 + \dots,\end{aligned}$$

satisfy

$$1 + \bar{G}(x) = \exp(\bar{C}(x)), \quad \bar{C}'(x) = \exp(x + \bar{B}'(x\bar{C}'(x)))$$

where $\bar{C}'(x)$ and $\bar{B}'(x)$ denote the derivatives of $\bar{C}(x)$ and $\bar{B}(x)$. The growth rate of $\{\bar{G}_n\}_{n=1}^{\infty}$, defined as $\gamma_l = \lim_{n \rightarrow \infty} (\bar{G}_n/n!)^{1/n}$, can be proved to exist and satisfies $27.22685 < \gamma_l < 27.22688$ [17, 18, 19]. It is known that $\gamma_l < \gamma_u$, hence the lower bound for γ_l serves as a lower bound for γ_u . Further, the asymptotics of $\{\bar{B}_n\}_{n=1}^{\infty}$ are exactly known [20]:

$$\bar{B}_n \sim \lambda n^{-7/2} \beta^n n!$$

where

$$\beta = \frac{16\tau^3}{(1+3\tau)(1-\tau)^3} = 26.1841125556\dots = (0.0381910976\dots)^{-1},$$

τ is the unique solution of

$$\frac{1+2t}{(1+3t)(1-t)} \exp\left[-\frac{t^2(1-t)(18+36t+5t^2)}{2(3+t)(1+2t)(1+3t)^2}\right] - 2 = 0$$

and λ is a constant (again omitted). The growth constant for $\{\bar{C}_n\}_{n=1}^{\infty}$ is clearly the same as that for $\{\bar{G}_n\}_{n=1}^{\infty}$, but which of the following formulas [19]:

$$\bar{C}_n \sim \mu n^{-5/2} \gamma_l^n n! \quad \text{or} \quad \bar{C}_n \sim \mu n^{-7/2} \gamma_l^n n!$$

is true? This appears to be a difficult question.

A graph is **outerplanar** if it can be embedded in the plane so that all its vertices lie on the same face. This face, by convention, is usually chosen to be the unbounded exterior of the graph. Any tree is an outerplanar graph. Non-outerplanar graphs include the complete graph K_4 with 4 vertices and all 6 edges, and the complete bipartite graph $K_{2,3}$ with 5 vertices and 6 edges. The unlabeled case has not received much attention, except in the 2-connected case (the first three graphs in Figure 5, each pentagonal, constitute all possibilities with 5 vertices). The generating function for unlabeled 2-connected outerplanar graphs [21]

$$\begin{aligned} \hat{b}(x) &= \sum_{n=1}^{\infty} \hat{b}_n x^n \\ &= x^3 + 2x^4 + 3x^5 + 9x^6 + 20x^7 + 75x^8 + 262x^9 + 1117x^{10} + \dots \end{aligned}$$

was obtained by Read [22, 23, 24], building on Motzkin [25]:

$$\begin{aligned} \hat{b}(x) &= \frac{(3x^2 - 2xf(x) + f(x)^2) - (2 + 2x + 7x^2 - 4xf(x) + 2f(x)^2)f(x^2) + 2f(x^2)^2}{4(2f(x^2) - 1)} \\ &\quad + \frac{1}{2} \sum_{k=3}^{\infty} \frac{1}{k} \left(\sum_{d|k} \varphi(d) (f(x^d))^{k/d} \right) \end{aligned}$$

where

$$f(x) = x + \sum_{r=1}^{\infty} \frac{1}{r} \sum_{s=2}^{\infty} \binom{s-2}{r-1} \binom{r+s-1}{s} x^s$$

and φ is the Euler totient function. Counting such graphs is closely related to enumerating the number of dissections of the interior of a regular n -gon into smaller polygons by use of nonintersecting diagonals. Asymptotics for \hat{b}_n are presently open. It would be good to learn more about \hat{c}_n and \hat{g}_n too.

In contrast, the generating function for labeled outerplanar graphs [21]

$$\begin{aligned} \hat{G}(x) &= \sum_{n=1}^{\infty} \frac{\hat{G}_n}{n!} x^n \\ &= x + \frac{2}{2!}x^2 + \frac{8}{3!}x^3 + \frac{63}{4!}x^4 + \frac{893}{5!}x^5 + \frac{19714}{6!}x^6 + \frac{597510}{7!}x^7 + \dots, \end{aligned}$$

the generating function for labeled connected outerplanar graphs

$$\begin{aligned} \hat{C}(x) &= \sum_{n=1}^{\infty} \frac{\hat{C}_n}{n!} x^n \\ &= x + \frac{1}{2!}x^2 + \frac{4}{3!}x^3 + \frac{37}{4!}x^4 + \frac{602}{5!}x^5 + \frac{14436}{6!}x^6 + \frac{458062}{7!}x^7 + \dots, \end{aligned}$$

and the generating function for labeled 2-connected outerplanar graphs

$$\begin{aligned} \hat{B}(x) &= \sum_{n=1}^{\infty} \frac{\hat{B}_n}{n!} x^n \\ &= \frac{1}{3!} x^3 + \frac{9}{4!} x^4 + \frac{132}{5!} x^5 + \frac{2700}{6!} x^6 + \frac{70920}{7!} x^7 + \frac{2275560}{8!} x^8 + \dots, \end{aligned}$$

satisfy

$$1 + \hat{G}(x) = \exp(\hat{C}(x)), \quad \hat{C}'(x) = \exp(x\hat{C}'(x) + \hat{B}'(x\hat{C}'(x)))$$

and, further,

$$\hat{B}'(x) = \frac{1}{8} \left(1 + 5x - \sqrt{1 - 6x + x^2} \right) - x.$$

In view of the algebraic nature of $\hat{B}'(x)$, it is not surprising that the growth constants possess closed-form expressions [26]:

$$\lim_{n \rightarrow \infty} \left(\frac{\hat{G}_n}{n!} \right)^{1/n} = \frac{1}{\xi} \exp \left(\frac{1 + 5\xi - \sqrt{1 - 6\xi + \xi^2}}{8} \right) = 7.3209800548\dots$$

where $\xi = 0.1707649868\dots$ has minimal polynomial $8 - 58x + 70x^2 - 28x^3 + 3x^4$, and

$$\lim_{n \rightarrow \infty} \left(\frac{\hat{B}_n}{n!} \right)^{1/n} = 3 + 2\sqrt{2} = 5.8284271247\dots$$

Like before, the growth constant for $\{\hat{C}_n\}_{n=1}^{\infty}$ is the same as that for $\{\hat{G}_n\}_{n=1}^{\infty}$.

Addendum. Giménez & Noy [27] have demonstrated that $\gamma_l = 27.2268777685\dots$; Bodirsky, Fusy, Kang & Vigerske [28] have shown that

$$\hat{g}_n \sim (0.00909941\dots)n^{-5/2}(7.50360\dots)^n,$$

with similar results for \hat{b}_n and \hat{c}_n .

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