

Erdős' Minimum Overlap Problem

STEVEN FINCH

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Let A, B be disjoint, complementary subsets of the set $\{1, 2, 3, \dots, 2n\}$ with cardinality $|A| = |B| = n$. Let M_k denote the number of solutions of the equation $a_i - b_j = k$, where k is an integer between $-2n$ and $2n$. Define

$$M(n) = \min_{A,B} \max_k M_k.$$

We wish to estimate $M(n)$ as n grows large [1, 2, 3]. The work of Erdős, Scherk and others [4, 5, 6] implies that

$$\mu_L = \liminf_{n \rightarrow \infty} \frac{M(n)}{n} \geq \sqrt{4 - \sqrt{15}} > 0.35639$$

and specific examples provide that [7]

$$\mu_R = \limsup_{n \rightarrow \infty} \frac{M(n)}{n} \leq \frac{2}{5} = 0.4.$$

Haugland [6] recently demonstrated that $\mu_L = \mu_R$ (meaning that the limit exists) and, using a theorem of Swinnerton-Dyer, obtained the improvement

$$0.35639 < \mu = \lim_{n \rightarrow \infty} \frac{M(n)}{n} < 0.38201.$$

No one has conjectured an exact value for this limiting ratio.

Observe that M_{-k} is the cardinality of the set $A_k \cap B$, where A_k is the translated set $\{a + k : a \in A\}$. Mycielski and Świerczkowski [4] considered a continuous analog of Erdős' problem. Let X, Y be disjoint, complementary measurable subsets of the interval $[0, 1]$ with Lebesgue measure $|X| = |Y| = 1/2$. It is not surprising that

$$\inf_{X,Y} \sup_t |X_t \cap Y| = \frac{\mu}{2}$$

where X_t is the translated set $\{x + t : x \in X\}$. Hence the bounds $0.17819 < \mu/2 < 0.19101$ carry over from before.

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Moser and Murdeshwar [8, 9, 10] studied the following generalization. Let f, g be Lebesgue integrable functions on \mathbb{R} satisfying

$$0 \leq f(x) \leq 1 \quad \text{for } 0 \leq x \leq 1, \quad f(x) = 0 \quad \text{otherwise};$$

$$0 \leq g(x) \leq 1 \quad \text{for } 0 \leq x \leq 1, \quad g(x) = 0 \quad \text{otherwise};$$

$$\int_0^1 f(x) dx = \frac{1}{2} = \int_0^1 g(x) dx.$$

(This scenario reduces to the preceding case by taking f to be the characteristic function of X and g to be the characteristic function of Y ; clearly $f(x) + g(x) = 1$ for all $0 \leq x \leq 1$.) Define

$$\lambda = \inf_{f,g} \sup_t \int_0^1 f(x+t) g(x) dx.$$

It is known [10] that $0.136 \leq \lambda \leq 0.166$, but it is not presently known whether Swinnerton-Dyer's theorem [6] can be applied here (in some extended form) to improve these bounds.

Here is a related problem due to Czipser [3, 11]. Let $\tilde{a}_1 < \tilde{a}_2 < \tilde{a}_3 < \dots < \tilde{a}_n$ be arbitrary integers and define $\tilde{A}_k = \{\tilde{a}_j + k : 1 \leq j \leq n\}$ for each integer k . Let \tilde{M}_k denote the cardinality $|\tilde{A}_k - \tilde{A}_0|$, that is, the number of elements of \tilde{A}_k not in \tilde{A}_0 . Define

$$\tilde{M}(n) = \min_A \max_{-n \leq k \leq n} \tilde{M}_k$$

and $\tilde{\mu}_L, \tilde{\mu}_R$ as earlier. It is known that $1/2 \leq \tilde{M}(n)/n \leq 2/3$ and, further, that $\tilde{M}(n)/n \geq 3/5$ for all $n \geq 26$ [12]. It is conjectured that $\tilde{\mu}_L = \tilde{\mu}_R = 2/3$. We give the corresponding functional version. Let \tilde{f} be a Lebesgue integrable function on \mathbb{R} satisfying

$$0 \leq \tilde{f}(x) \leq 1, \quad \int_{-\infty}^{\infty} \tilde{f}(x) dx = 1.$$

Define

$$\tilde{\lambda} = \inf_{\tilde{f}} \left\{ 1 - \inf_{-1 \leq t \leq 1} \int_{-\infty}^{\infty} \tilde{f}(x+t) \tilde{f}(x) dx \right\}.$$

It is known that $0.5892 \leq \tilde{\lambda} \leq 2/3$ [11]. As a corollary, if \tilde{X} is a measurable subset of \mathbb{R} with Lebesgue measure $|\tilde{X}| = 1$, then

$$0.5892 \leq \inf_{\tilde{X}} \sup_{-1 \leq t \leq 1} |\tilde{X}_t - \tilde{X}| \leq \frac{2}{3}.$$

The discrete and continuous analogs do not appear to be as closely linked as before. Again, we wonder whether recent techniques [6] can be invoked to sharpen these bounds.

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