

Ornstein-Uhlenbeck Process

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We first define several words. A stochastic process $\{Y_t : t \geq 0\}$ is

- **stationary** if, for all $t_1 < t_2 < \dots < t_n$ and $h > 0$, the random n -vectors $(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n})$ and $(Y_{t_1+h}, Y_{t_2+h}, \dots, Y_{t_n+h})$ are identically distributed; that is, time shifts leave joint probabilities unchanged
- **Gaussian** if, for all $t_1 < t_2 < \dots < t_n$, the n -vector $(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n})$ is multivariate normally distributed
- **Markovian** if, for all $t_1 < t_2 < \dots < t_n$, $P(Y_{t_n} \leq y | Y_{t_1}, Y_{t_2}, \dots, Y_{t_{n-1}}) = P(Y_{t_n} \leq y | Y_{t_{n-1}})$; that is, the future is determined only by the present and not the past.

Also, a process $\{Y_t : t \geq 0\}$ is said to have **independent increments** if, for all $t_0 < t_1 < \dots < t_n$, the n random variables $Y_{t_1} - Y_{t_0}, Y_{t_2} - Y_{t_1}, \dots, Y_{t_n} - Y_{t_{n-1}}$ are independent. This condition implies that $\{Y_t : t \geq 0\}$ is Markovian, but not conversely. The increments are further said to be **stationary** if, for any $t > s$ and $h > 0$, the distribution of $Y_{t+h} - Y_{s+h}$ is the same as the distribution of $Y_t - Y_s$. This additional provision is needed for the following definition.

A stochastic process $\{W_t : t \geq 0\}$ is a **Wiener-Lévy process** or **Brownian motion** if it has stationary independent increments, if W_t is normally distributed and $E(W_t) = 0$ for each $t > 0$, and if $W_0 = 0$. It follows immediately that $\{W_t : t > 0\}$ is Gaussian and that $\text{Cov}(W_s, W_t) = \theta^2 \min\{s, t\}$, where the variance parameter θ^2 is a positive constant. For concreteness' sake, we henceforth assume that $\theta = 1$. Almost all sample paths of Brownian motion are everywhere continuous but nowhere differentiable.

One technical stipulation is required for the following. A stochastic process $\{Y_t : t \geq 0\}$ is **continuous in probability** if, for all $u \in \mathbb{R}^+$ and $\varepsilon > 0$, $P(|Y_v - Y_u| \geq \varepsilon) \rightarrow 0$ as $v \rightarrow u$. This holds if $\text{Cov}(Y_s, Y_t)$ is continuous over $\mathbb{R}^+ \times \mathbb{R}^+$. Note that this is a statement about distributions, not sample paths.

Having dispensed with preliminaries, we turn to the central topic. A stochastic process $\{X_t : t \geq 0\}$ is an **Ornstein-Uhlenbeck process** or a **Gauss-Markov**

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process if it is stationary, Gaussian, Markovian, and continuous in probability [1, 2]. A fundamental theorem, due to Doob [3, 4, 5], ensures that $\{X_t : t \geq 0\}$ necessarily satisfies the following linear stochastic differential equation:

$$dX_t = -\rho(X_t - \mu)dt + \sigma dW_t$$

where $\{W_t : t \geq 0\}$ is Brownian motion with unit variance parameter and μ, ρ, σ are constants. We have moments

$$\mathbb{E}(X_t) = \mu, \quad \text{Cov}(X_s, X_t) = \frac{\sigma^2}{2\rho} e^{-\rho|s-t|}$$

in the unconditional (strictly stationary) case and

$$\begin{aligned} \mathbb{E}(X_t | X_0 = c) &= \mu + (c - \mu)e^{-\rho t} \\ \text{Cov}(X_s, X_t | X_0 = c) &= \frac{\sigma^2}{2\rho} \left(e^{-\rho|s-t|} - e^{-\rho(s+t)} \right) \end{aligned}$$

in the conditional (asymptotically stationary) case, where X_0 is initially constant. The latter case encompasses Brownian motion when $\mu = c = 0$, $\sigma = 1$ and $\rho \rightarrow 0^+$. The former case encompasses idealized **white noise** $\{dW_t/dt : t \geq 0\}$ when $\mu = 0$, $\sigma = \rho$ and $\rho \rightarrow \infty$.

Before proceeding, we note the following simple algorithm for generating a sample path of the Ornstein-Uhlenbeck process (also known as **colored noise**) over the time interval $[0, T]$. Let N be a large integer and let z_0, z_1, \dots, z_N be independent random numbers generated from a normal distribution with mean 0 and variance $\sigma^2/(2\rho)$. Define $x_0 = \mu + z_0$ for the unconditional case and $x_0 = c$ for the conditional case. Then define recursively

$$x_n = \mu + \kappa_N(x_{n-1} - \mu) + \sqrt{1 - \kappa_N^2} z_n$$

for $1 \leq n \leq N$, where $\kappa_N = \exp(-\rho T/N)$. The sequence x_0, x_1, \dots, x_N is called a first-order autoregressive sequence (a discrete analog of the OU process) with lag-one correlation coefficient κ_N . Finally, interpolate linearly the values $X(nT/N) = x_n$ for $0 \leq n \leq N$ to obtain the desired path [6, 7, 8]. More sophisticated simulation methods are found in [9, 10, 11].

For concreteness' sake, we henceforth assume that $\mu = 0$, $\rho = 1$ and $\sigma^2 = 2$. (Some authors take $\sigma^2 = 1$ instead; the decision becomes apparent in any paper by seeing whether $\text{Cov}(X_s, X_t)$ is $e^{-|s-t|}$ or $e^{-|s-t|}/2$.) The conditional probability

$$\mathbb{P}(X_t \leq x | X_0 = c) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \int_{-\infty}^x \exp\left(-\frac{(\xi - ce^{-t})^2}{2(1 - e^{-2t})}\right) d\xi$$

tends to the standard normal distribution, of course, as $t \rightarrow \infty$ (meaning that transients die out with time and don't affect long-term behavior). Likewise, $P(X_s \leq x \text{ and } X_t \leq y | X_0 = c)$ can be evaluated. One might believe that the solution of any problem involving the OU process would be similarly straightforward; the following sections serve, however, to eliminate such ideas [12, 13].

0.1. First-Passage Times. For $a \in \mathbb{R}$, we wish to find the length of time required for an OU process to cross the level $x = a$, given that it started at $x = c$. Define the **first-passage time** or **hitting time** $T_{a,c}$ by $T_{a,c} = \inf \{t \geq 0 : X_t = a | X_0 = c\}$. The random variable $T_{a,c}$ is 0 if and only if $a = c$. Let $f_{a,c}(t)$ denote the density function of $T_{a,c}$. In the special case when $a = 0$, it is known that [2, 12, 14, 15]

$$f_{0,c}(t) = \sqrt{\frac{2}{\pi}} \frac{|c|e^{-t}}{(1 - e^{-2t})^{3/2}} \exp\left(-\frac{c^2 e^{-2t}}{2(1 - e^{-2t})}\right)$$

but for $a \neq 0$, the formulas for $f_{a,c}(t)$ are more complicated (as we shall soon see). For $a > 0$ and $c > 0$, Thomas [16] and Ricciardi & Sato [17, 18] demonstrated that

$$\begin{aligned} E(T_{a,0}) &= \sqrt{\frac{\pi}{2}} \int_0^a \left(1 + \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right)\right) \exp\left(\frac{t^2}{2}\right) dt = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \Gamma\left(\frac{k}{2}\right), \\ E(T_{0,c}) &= \sqrt{\frac{\pi}{2}} \int_{-c}^0 \left(1 + \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right)\right) \exp\left(\frac{t^2}{2}\right) dt = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\sqrt{2}c)^k}{k!} \Gamma\left(\frac{k}{2}\right) \end{aligned}$$

and, for example,

$$\begin{aligned} E(T_{1,0}) &= 2.0934066496\dots, & E(T_{0,1}) &= 0.9019080126\dots, \\ E(T_{2,0}) &= 10.4284093979\dots, & E(T_{0,2}) &= 1.4252045655\dots \end{aligned}$$

The asymmetry in going from 0 to x , versus going from x to 0, is unsurprising: The process has mean 0, hence it tends to arrive at 0 more often than it departs from 0. For $a > 0$ and $c > 0$, we have [17, 18]

$$\begin{aligned} \operatorname{Var}(T_{a,0}) &= \sqrt{2\pi} \int_0^a \int_{-\infty}^t \int_s^a \left(1 + \operatorname{erf}\left(\frac{r}{\sqrt{2}}\right)\right) \exp\left(\frac{r^2 + t^2 - s^2}{2}\right) dr ds dt - E(T_{a,0})^2 \\ &= E(T_{a,0})^2 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \Gamma\left(\frac{k}{2}\right) \Psi\left(\frac{k}{2}\right), \end{aligned}$$

$$\begin{aligned} \text{Var}(T_{0,c}) &= \sqrt{2\pi} \int_{-c}^0 \int_{-\infty}^t \int_s^0 \left(1 + \text{erf}\left(\frac{r}{\sqrt{2}}\right)\right) \exp\left(\frac{r^2 + t^2 - s^2}{2}\right) dr ds dt - \mathbb{E}(T_{0,c})^2 \\ &= \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{(\sqrt{2}c)^k}{k!} \Gamma\left(\frac{k}{2}\right) \Psi\left(\frac{k}{2}\right) - \mathbb{E}(T_{0,c})^2 \end{aligned}$$

where $\Psi(x) = \psi(x) - \psi(1)$ and $\psi(x)$ is the digamma function [19]. In particular, $\Psi(1) = 0$ and

$$\Psi(x) = \begin{cases} \sum_{j=1}^{x-1} \frac{1}{j} & \text{if } x \text{ is an integer } > 1 \\ -2\ln(2) + 2 \sum_{j=1}^{x-1/2} \frac{1}{2j-1} & \text{if } x \text{ is a half-integer } > 0. \end{cases}$$

For example,

$$\begin{aligned} \text{Var}(T_{1,0}) &= 5.8420278024\dots, & \text{Var}(T_{0,1}) &= 0.8510837032\dots, \\ \text{Var}(T_{2,0}) &= 105.2752035488\dots, & \text{Var}(T_{0,2}) &= 1.0669454393\dots \end{aligned}$$

To compute $f_{a,c}(t)$ exactly for arbitrary a and c , we would need to invert the following (Laplace transform) identity due to Darling & Siegert [20, 21, 22, 23]:

$$\mathbb{E}(e^{-\lambda T_{a,c}}) = \int_0^{\infty} f_{a,c}(t) e^{-\lambda t} dt = \begin{cases} \frac{D_{-\lambda}(-c)}{D_{-\lambda}(-a)} \exp\left(\frac{c^2 - a^2}{4}\right) & \text{if } c < a \\ \frac{D_{-\lambda}(c)}{D_{-\lambda}(a)} \exp\left(\frac{c^2 - a^2}{4}\right) & \text{if } c > a \end{cases}$$

where $D_{\nu}(x)$ is the **parabolic cylinder function** or **Weber function** [24]:

$$D_{\nu}(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \exp\left(\frac{x^2}{4}\right) \int_0^{\infty} t^{\nu} \exp\left(-\frac{t^2}{2}\right) \cos\left(xt - \frac{\nu\pi}{2}\right) dt & \text{if } \nu > -1 \\ \frac{1}{\Gamma(-\nu)} \exp\left(-\frac{x^2}{4}\right) \int_0^{\infty} t^{-\nu-1} \exp\left(-\frac{t^2}{2} - xt\right) dt & \text{if } \nu < 0. \end{cases}$$

The two branches of this formula agree for $-1 < \nu < 0$. A differential equation

$$\frac{d^2 y}{dx^2} - \left(\frac{x^2}{4} - \nu - \frac{1}{2}\right) y(x) = 0$$

is satisfied by $D_{\nu}(x)$ and, if ν is not an integer, independently by $D_{-\nu}(x)$. A series representation in terms of confluent hypergeometric functions [0.4] is also useful.

Unfortunately a closed-form expression for the inverse Laplace transform seems not to be possible; only a numerical approach is feasible at present. Keilson & Ross [25] tabulated the distribution of $T_{a,c}$ for a number of values a and c . For example, the median time for an OU process X_t to reach $a = 1$, given that $X_0 = c = 0$, is 1.1892.... This corresponds to the 50th percentile of the distribution of $T_{1,0}$. The median of $T_{2,0}$, by contrast, is 7.2521....

We turn to a more complicated problem involving two (absorbing) boundaries rather than just one. Given $a < c < b$, what is the length of time required for the process to escape the interval (a, b) , given that it started at $x = c$? Define $T_{a,b,c} = \inf \{t \geq 0 : X_t = a \text{ or } X_t = b \mid X_0 = c\}$ and let $f_{a,b,c}(t)$ denote the density function of $T_{a,b,c}$. Efforts have focused on the scenario in which $-a = b > 0$. The Laplace transform of $f_{-b,b,c}(t)$ satisfies [22]

$$\mathbb{E}(e^{-\lambda T_{-b,b,c}}) = \frac{D_{-\lambda}(c) + D_{-\lambda}(-c)}{D_{-\lambda}(b) + D_{-\lambda}(-b)} \exp\left(\frac{c^2 - b^2}{4}\right)$$

assuming $-b < c < b$. From another table in [25], the median of $T_{-1,1,0}$ is found to be 0.4449.... The reason that this is less than 1.1892... is clear: Each direction of travel leads to a potential crossing. The median of $T_{-2,2,0}$ is 3.2439....

Keilson & Ross' approach to evaluating such probabilities was based on finding zeroes and residues in the complex plane of the parabolic cylinder functions. Alternative approaches for numerically computing $f_{a,c}(t)$ and $f_{-b,b,c}(t)$ include [26, 27, 28, 29]. We report on some related asymptotics in [0.4].

There is an obvious connection between first-passage times and extreme values of a process (in the conditional case). We simply summarize:

$$\left. \begin{array}{l} \mathbb{P}\left(\max_{0 \leq t \leq T} X_t \leq a \mid X_0 = c\right) \quad \text{if } c < a \\ \mathbb{P}\left(\min_{0 \leq t \leq T} X_t \geq a \mid X_0 = c\right) \quad \text{if } c > a \end{array} \right\} = \mathbb{P}(T_{a,c} > T) = 1 - F_{a,c}(T)$$

and, if $a < c < b$,

$$\mathbb{P}\left(a \leq \min_{0 \leq t \leq T} X_t \leq \max_{0 \leq t \leq T} X_t \leq b \mid X_0 = c\right) = \mathbb{P}(T_{a,b,c} > T) = 1 - F_{a,b,c}(T)$$

where $F_{a,c}(t)$, $F_{a,b,c}(t)$ are the cumulative distribution functions of $T_{a,c}$, $T_{a,b,c}$. In the special case when $-a = b > 0$, the latter formula becomes a statement about $\max_{0 \leq t \leq T} |X_t|$, given $X_0 = c$. Also, the **range** of the process satisfies [22]

$$\mathbb{P}\left(\max_{0 \leq t \leq T} X_t - \min_{0 \leq t \leq T} X_t \leq r \mid X_0 = c\right) = \int_0^r \int_{c-q}^c \frac{\partial^2}{\partial a \partial b} F_{a,b,c}(T) \Big|_{b=a+q} da dq$$

but no one apparently has calculated this probability.

0.2. Historical Maximums. If the condition $X_0 = c$ is discarded, what then can be said about $\max_{0 \leq t \leq T} X_t$ or $\max_{0 \leq t \leq T} |X_t|$? We focus solely on the former expression and write $M_T = \max_{0 \leq t \leq T} X_t$. It can be shown that [30, 31, 32]

$$P(M_T \leq 0) = \frac{1}{\pi} \arcsin(e^{-T})$$

which is a beautiful (but isolated) result. More generally [32],

$$\int_0^\infty P(M_t \leq y) e^{-\lambda t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \frac{1}{\lambda} \left(1 - \frac{D_{-\lambda}(-x)}{D_{-\lambda}(-y)} \exp\left(\frac{x^2 - y^2}{4}\right) \right) \exp\left(-\frac{x^2}{2}\right) dx$$

for arbitrary y , or

$$\int_0^\infty g_t(y) e^{-\lambda t} dt = \frac{1}{\sqrt{2\pi}} \frac{D_{-\lambda-1}(-y)^2}{D_{-\lambda}(-y)^2} \exp\left(\frac{-y^2}{2}\right)$$

where $g_t(y)$ is the density function of M_t . For example, the median value of M_1 is 1.0393... and the median value of M_{10} is 2.2202.... It can be inferred from [0.3] that the median of M_T is asymptotically $\sqrt{2 \ln(T)}$ as $T \rightarrow \infty$.

An alternative approach for numerically computing $P(M_t \leq y)$ via the Mellin transform is due to DeLong [33, 34, 35]. We hope to report on this later. An interesting application to computer science, involving the maximum size reached by a dynamic data structure over a long span of time, is described in [36].

0.3. Pickands' Constants. Assume that $\{Y_t : t \geq 0\}$ is a stationary Gaussian process with zero mean, unit variance and covariance function of the form

$$r(|s - t|) = \text{Cov}(Y_s, Y_t) = 1 - C |s - t|^\alpha + o(|s - t|^\alpha)$$

as $|s - t| \rightarrow 0$, where $0 < \alpha \leq 2$ and $C > 0$ are constants. Assume further that $r(\tau) \ln(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. Pickands [37, 38, 39, 40, 41] demonstrated that $M_T = \max_{0 \leq t \leq T} Y_t$ has the Gumbel limiting distribution [42]

$$\lim_{T \rightarrow \infty} P\left(\sqrt{2 \ln(T)} (M_T - k_T) \leq x\right) = \exp(-e^{-x})$$

where

$$k_T = \sqrt{2 \ln(T)} + \frac{1}{\sqrt{2 \ln(T)}} \left\{ \frac{2 - \alpha}{2\alpha} \ln(\ln(T)) + \ln\left((2\pi)^{-\frac{1}{2}} 2^{\frac{2-\alpha}{2\alpha}} C^{\frac{1}{\alpha}} H_\alpha\right) \right\}$$

and H_α is a positive constant independent of C . It is known that $H_1 = 1$ (corresponding to the OU process) and $H_2 = 1/\sqrt{\pi}$. No other exact values for H_α are known. An alternative characterization of H_α is

$$H_\alpha = \lim_{T \rightarrow \infty} \int_0^\infty \mathbb{P}(\tilde{M}_T > y) e^y dy$$

where $\{\tilde{Y}_t : t \geq 0\}$ is a nonstationary Gaussian process with

$$\mathbb{E}(\tilde{Y}_t) = -|t|^\alpha, \quad \text{Cov}(\tilde{Y}_s, \tilde{Y}_t) = |s|^\alpha + |t|^\alpha - |s - t|^\alpha$$

but this does not seem to help. Shao [43] and Debicki, Michna & Rolski [44] gave bounds on H_α ; for example,

$$0.009 \leq H_{1/2} \leq 715.94, \quad 0.208 \leq H_{3/2} \leq 3.04.$$

A conjecture that $H_\alpha = 1/\Gamma(1/\alpha)$ remains unproved. There is also a connection with the Gaussian correlation conjecture and with estimating small ball probabilities [45], topics which we hope to address later.

0.4. Upper Tail Asymptotics. We revisit the single-boundary first-passage time distribution and ask about the limiting value

$$\lambda(a) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \{\mathbb{P}(T_{a,0} > t)\}$$

as a function of $a > 0$. In words, what can be said about the upper tail of the distribution of the first hitting time $T_{a,0}$ for an OU process X_t across the level $x = a$, given that $X_0 = 0$? Mandl [46, 47] and Beekman [48] demonstrated that $-1 < \lambda(a) < 0$ and that $\lambda(a)$ is the zero of $D_{-\lambda}(-a)$ closest to 0. Sample values include [17, 49, 50]

$$\lim_{a \rightarrow 0^+} \lambda(a) = -1, \quad \lim_{a \rightarrow \infty} \lambda(a) \cdot \frac{\exp(a^2/2)}{a} = \frac{-1}{\sqrt{2\pi}},$$

$$\lambda(0.7649508673\dots) = -\frac{1}{2},$$

$$\lambda(1) = -0.3882382947\dots = 2(-0.1941191473\dots),$$

$$\lambda(2) = -0.0972745958\dots = 2(-0.0486372979\dots).$$

For the symmetric double-boundary first-passage time distribution, we examine

$$\lambda(-b, b) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \{\mathbb{P}(T_{-b,b,0} > t)\}$$

as a function of $b > 0$. Breiman [51] proved that $-\infty < \lambda(-b, b) < 0$ and that $\lambda(-b, b)$ is the zero of $\Phi(\lambda/2, 1/2, b^2/2)$ closest to 0, where

$$\Phi(u, v, w) = 1 + \sum_{k=1}^{\infty} \frac{u(u+1)(u+2)\cdots(u+k-1)}{v(v+1)(v+2)\cdots(v+k-1)} \frac{w^k}{k!}$$

is the **confluent hypergeometric function of the first kind**. For simplicity, define $\mu(b) = \lambda(-b, b)$. Sample values include [50, 51, 52]

$$\begin{aligned} \lim_{b \rightarrow 0^+} \mu(b) &= -\infty, & \lim_{b \rightarrow \infty} \mu(b) \cdot \frac{\exp(b^2/2)}{b} &= \frac{-1}{\sqrt{2\pi}}, \\ \mu(1) &= -2, & \mu(1.3069297277\dots) &= -1, & \mu(1.6438001904\dots) &= -\frac{1}{2}, \\ \mu\left(\sqrt{3 - \sqrt{6}}\right) &= \mu(0.7419637843\dots) &= -4, \\ \mu(2) &= -0.2429928807\dots, & \mu(3) &= -0.0239463006\dots, \\ \mu(\sqrt{2}) &= -0.7984598320\dots, & \mu(2\sqrt{2}) &= -0.0374612092\dots \end{aligned}$$

The latter two values come from [52], where a different time scaling was chosen. Also, the constant $(3 - 6^{1/2})^{1/2}$ appears in [53, 54, 55] with regard to stopping rules in statistical sequential analysis.

For completeness' sake, here is the expression for $D_{-\lambda}(x)$ in terms of confluent hypergeometric functions:

$$D_{-\lambda}(x) = \frac{\sqrt{\pi}2^{-\lambda/2}}{\Gamma((1+\lambda)/2)} e^{-x^2/4} \Phi\left(\frac{\lambda}{2}, \frac{1}{2}, \frac{x^2}{2}\right) - 2 \frac{\sqrt{\pi}2^{-(1+\lambda)/2}}{\Gamma(\lambda/2)} x e^{-x^2/4} \Phi\left(\frac{1+\lambda}{2}, \frac{3}{2}, \frac{x^2}{2}\right)$$

which gives rise to the values $\lambda(1)$, $\lambda(2)$ and $\lambda^{-1}(-1/2)$ listed earlier. The constant $\mu^{-1}(-1)$ is important in the study of sample path behavior of Brownian motion [50, 56, 57] and first appeared in [54], as far as is known. Some higher dimensional results are given in [50, 58]. Csáki [59, 60] recently outlined the distributional asymptotics of the maximum M_T , but we cannot discuss this topic further.

0.5. Addendum. New numerical transform inversion algorithms [61, 62, 63] make enhancement of the tables in [25, 32] possible. Also, the distribution of the L_2 -norm of X_t on $[0, T]$ can be inferred from closed-form expressions in [64, 65]. We wonder about corresponding results for L_1 and L_∞ -norms.

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