Nash's Inequality

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Consider all smooth, compactly supported, s-integrable functions $f:\mathbb{R}^n\rightarrow\mathbb{R}$ with the property that the Euclidean norm of the gradient $\nabla f:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is q-integrable:

$$||f||_s=\left(\int\limits_{\mathbb{R}^n}|f(x)|^s\,dx\right)^{\frac{1}{s}}<\infty,\qquad s\geq 1;$$

$$||\nabla f||_q=\left(\int\limits_{\mathbb{R}^n}|\nabla f(x)|^q\,dx\right)^{\frac{1}{q}}<\infty,\qquad q\geq 1.$$

For example, let $q=2$ and $s=1$. Nash's inequality [1]

$$||f||_2^{2+\frac{4}{n}}\leq A_n\cdot||\nabla f||_2^2\cdot||f||_1^{\frac{4}{n}},$$

that is,

$$\left(\int\limits_{\mathbb{R}^n}|f(x)|^2\,dx\right)^{1+\frac{2}{n}}\leq A_n\left(\int\limits_{\mathbb{R}^n}|\nabla f(x)|^2\,dx\right)\left(\int\limits_{\mathbb{R}^n}|f(x)|\,dx\right)^{\frac{4}{n}},$$

is useful in the study of nonlinear partial differential equations (PDEs). Best constants A_n were proved by Carlen & Loss [2] to be

$$A_n=\left(1+\frac{2}{n}\right)\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}\frac{1}{\pi j_{n/2,1}^2}$$

where $j_{n/2,1}$ is the smallest positive zero [3] of the Bessel function $J_{n/2}(x)$. Hence

$$A_1=\frac{27}{16\pi^2}=0.1709794973...,\qquad A_2=0.0867212975...,\qquad A_3=0.0585146159...$$

and $A_n\sim 2/(\pi e n)$ as $n\rightarrow\infty$. This asymptotic result is due to Beckner [4, 5, 6, 7].

As another example, let $q=2$ and $s=2$. Best constants for Moser's inequality [8, 9]

$$||f||_{2+\frac{4}{n}}^{2+\frac{4}{n}}\leq B_n\cdot||\nabla f||_2^2\cdot||f||_2^{\frac{4}{n}},$$

that is,

$$\int_{\mathbb{R}^n} |f(x)|^{2+\frac{4}{n}}\,dx \le B_n \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\,dx\right)\left(\int_{\mathbb{R}^n} |f(x)|^2\,dx\right)^{\frac{2}{n}},$$

are known exactly only for $n = 1$ [10]:

$$B_1 = \frac{4}{\pi^2} = 0.4052847345...$$

When $n = 2$, we have a numerical estimate $B_2 = 0.170927... = (5.85043...)^{-1}$; more will be said about this constant shortly. Here too it is known that $B_n \sim 2/(\pi e n)$ as $n \to \infty$ [5].

0.1. Gagliardo-Nirenberg. A generalization of Nash's inequality is [11, 12, 13]

$$||f||_r \le \kappa_n(q,r,s) \cdot ||\nabla f||_q^\theta \cdot ||f||_s^{1-\theta}$$

where $1 < q < n$, $s \ge 1$, $0 \le \theta \le 1$ and

$$\frac{1}{r} = \left(\frac{1}{q} - \frac{1}{n}\right)\theta + \frac{1}{s}(1-\theta).$$

These conditions force $r \ge 1$. Note that the Gagliardo-Nirenberg inequality trivially encompasses the p-Sobolev inequality when $q = p$ and $\theta = 1$ (details appear in [0.3]). We have already examined best constants for one case:

$$\kappa_n(2,2,1) = A_n^{\theta/2}, \qquad \theta = \frac{n}{n+2}$$

and wonder about any other nontrivial cases possessing explicit formulas for all n. Del Pino & Dolbeault recently discovered two one-parameter families that assist in answering the question [14, 15, 16, 17]:

$$\kappa_n\left(q, q\frac{s-1}{q-1}, s\right) = \left(\frac{s-q}{q\sqrt{\pi}}\right)^\theta \left(\frac{qs}{n(s-q)}\right)^{\theta/q} \left(\frac{\delta}{qs}\right)^{1/r} \times \left(\frac{\Gamma\left(s\frac{q-1}{s-q}\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{q-1}{q}\frac{\delta}{s-q}\right)\Gamma\left(n\frac{q-1}{q}+1\right)}\right)^{\theta/n}$$

for all $1 < q < s$, where $q(s-1) = r(q-1)$ and $\delta = nq - s(n-q) \ge q$, and

$$\kappa_n\left(q, r, q\frac{r-1}{q-1}\right) = \left(\frac{q-r}{q\sqrt{\pi}}\right)^\theta \left(\frac{qr}{n(q-r)}\right)^{\theta/q} \left(\frac{qr}{\delta}\right)^{(1-\theta)/s} \times \left(\frac{\Gamma\left(\frac{q-1}{q}\frac{\delta}{q-r}+1\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(r\frac{q-1}{q-r}+1\right)\Gamma\left(n\frac{q-1}{q}+1\right)}\right)^{\theta/n}$$

for all $1 < r < q$, where $q(r-1) = s(q-1)$ and $\delta = nq - r(n-q) > 0$.

Most cases, however, are like

$$\kappa_n(2, 2+\tfrac{4}{n}, 2) = B_n^{\theta/2}, \qquad \theta = \frac{n}{n+2}$$

in the sense that explicit expressions are presently unavailable for all n. For example [18, 19, 20, 21, 22],

$$\kappa_2(2,3,2) = \frac{1}{1.379427...}, \qquad \theta = \frac{1}{3};$$

$$\kappa_2(2,4,2) = B_2^{1/4} = \sqrt[4]{\frac{1}{\pi \cdot 1.86225...}} = \frac{1}{1.555239...}, \qquad \theta = \frac{1}{2};$$

$$\kappa_2(2,6,2) = \sqrt[3]{\frac{1}{4.5981...}} = \frac{1}{1.663066...}, \qquad \theta = \frac{2}{3};$$

$$\kappa_3(2,4,2) = \frac{1}{2.2258...}, \qquad \theta = \frac{3}{4}.$$

As a prelude to the next section, define

$$C_n(\sigma) = \kappa_n(2, 2\sigma+2, 2)$$

for $\sigma > 0$; this two-parameter family includes the four constants just listed.

0.2. Schrödinger. Let $\triangle$ denote the Laplacian operator. A space function $f(x)$ is **radial** if f is a function of $|x|$ alone. Also, a time function $g(t)$ is **global** if it is finite for all t, that is, no blow ups occur in finite time.

Here is an alternative characterization [19] of $C_n(\sigma)$ for $0 < \sigma < 2/(n-2)$:

$$C_n(\sigma) = \left(\frac{\sigma+1}{||\psi||_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth, positive, radial solution of the nonlinear PDE

$$\frac{n\sigma}{2}\triangle\psi - \frac{2\sigma+2-n\sigma}{2}\psi + \psi^{2\sigma+1} = 0$$

of minimal norm $||\psi||_2$ (the ground state). If $n = 2$, such a function $\psi(x)$ can be proved to be unique; further,

$$||\psi||_2^2 = (2\pi)(1.86225...)$$

when $\sigma = 1$. This gives rise to our numerical estimate of $C_2(1) = B_2^{1/4}$. It is known (among many things) that the cubic Schrödinger PDE in $\mathbb{R}^2$:

$$2i\frac{\partial\varphi}{\partial t} + \triangle\varphi + |\varphi|^2\varphi = 0$$

with initial conditions

$$\varphi(x,0)=\varphi_0(x)$$

possesses a global solution $\varphi:\mathbb{R}^2\times\mathbb{R}^+\to\mathbb{C}$ if $||\varphi_0||_2<||\psi||_2$. The latter inequality is sharp in a certain technical sense involving instability. Solutions $\varphi(x,t)$ of the Schrödinger equation find application in optics and plasma physics [23].

The constant B_2 also appears in the study of intersection local times for planar random walks and planar Brownian motion [24, 25, 26].

0.3. Sobolev. Let $\omega_n=\pi^{n/2}/\Gamma(n/2+1)$ denote the volume enclosed by the unit sphere in $\mathbb{R}^n$; consequently $\tilde{\omega}_{n-1}=n\omega_n$ is its surface area. For any $1\leq p<n$, let $p^*=np/(n-p)$. The classical p-Sobolev inequality is as follows:

$$\left(\int\limits_{\mathbb{R}^n}|f(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}}\leq K\left(\int\limits_{\mathbb{R}^n}|\nabla f(x)|^{p}\,dx\right)^{\frac{1}{p}}$$

and the best constant

$$K(n,p)=\kappa_n(p,p^*,s)\qquad(s\text{ is immaterial since }\theta=1)$$

was independently determined by Aubin [27, 28] and Talenti [29]:

$$\begin{aligned}K(n,p)&=\begin{cases}\dfrac{1}{n}\left(\dfrac{n}{\tilde{\omega}_{n-1}}\right)^{\frac{1}{n}} & \text{if } p=1\\ n^{-\frac{1}{p}}\left(\dfrac{p-1}{n-p}\right)^{1-\frac{1}{p}}\left(\dfrac{\Gamma(n+1)}{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})\tilde{\omega}_{n-1}}\right)^{\frac{1}{n}} & \text{if } 1<p<n\end{cases}\\ &=\begin{cases}\dfrac{1}{n}\left(\dfrac{1}{\omega_n}\right)^{\frac{1}{n}} & \text{if } p=1\\ n^{-\frac{1}{p}}\left(\dfrac{p-1}{n-p}\right)^{1-\frac{1}{p}}\left(\dfrac{\Gamma(n)}{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})\omega_n}\right)^{\frac{1}{n}} & \text{if } 1<p<n\end{cases}.\end{aligned}$$

Note the special case

$$K(n,2)=\sqrt{\frac{4}{n(n-2)\tilde{\omega}_n^{2/n}}}=(\pi n(n-2))^{-\frac{1}{2}}\left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{\frac{1}{n}}$$

which arises frequently in applications [30, 31]. Only the case $p=1$ was discussed in [32].

As an aside, let $p^{\#} = pn/(n-2p)$. The best constant in the second-order Sobolev inequality

$$\left(\int\limits_{\mathbb{R}^n} |f(x)|^{2^{\#}}\, dx\right)^{\frac{1}{2^{\#}}} \le M \left(\int\limits_{\mathbb{R}^n} |\triangle f(x)|^2\, dx\right)^{\frac{1}{2}}$$

is known to be [33, 34]

$$M(n) = \sqrt{\frac{16}{n(n-4)(n^2-4)\tilde{\omega}_{n+1}^{4/n}}} = \left(\pi^2 n(n-4)(n^2-4)\right)^{-\frac{1}{2}} \left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{\frac{2}{n}}$$

The similarity between $K(n,2,)$ and $M(n)$ is interesting: The former involves ∇f while the latter involves $\triangle f$. We wonder about the p-generalization of the latter.

0.4. Trudinger-Moser. A limiting scenario (as $p \to n^-$) of the Sobolev inequality is as follows. Let D denote a bounded open domain with smooth boundary in $\mathbb{R}^n$; for example, let D be an open ball. Let $|D|$ denote the Lebesgue measure of D. Consider all smooth, compactly supported functions $f : D \to \mathbb{R}$ with the property that ∇f is n-integrable and

$$\int\limits_D |\nabla f(x)|^n\, dx \le 1.$$

Then there exists a constant c_n depending only on n (and not on D) such that [35, 36]

$$\frac{1}{|D|}\int\limits_D \exp\left(\alpha \cdot |f(x)|^{n/(n-1)}\right) dx \le c_n$$

for any value $\alpha \le n\tilde{\omega}_{n-1}^{1/(n-1)}$. Further, if α exceeds the indicated threshold, then the left hand side can be made arbitrarily large by appropriate choice of $f(x)$.

Carleson & Chang [37] obtained that $c_2 = 4.3556...$ (with computational help by Gamelin). In principle, accurate estimates of c_n are possible, but no one appears to have done this. Variations and elaborations of the fascinating Trudinger-Moser inequality are found in [38, 39, 40, 41, 42, 43].

Addendum. Gunson [44] stated the first result of Del Pino & Dolbeault (in which q, s are free and $r = q(s-1)/(q-1)$), but without proof.

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