Minkowski-Siegel Mass Constants

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Let X denote either a vector space over the real numbers $\mathbb{R}$ or a module over the integers $\mathbb{Z}$. A symmetric positive definite bilinear form f on X is an **inner product** if, for any linear form g on X, there exists a unique $x \in X$ such that $g(y) = f(x, y)$ for all $y \in X$. This nondegeneracy condition is superfluous when X is a finite-dimensional vector space [1, 2]. The pair (X, f) is called an **inner product space** or an **inner product module**, respectively. Two pairs (X, f) and (X', f') are **isomorphic** if there is a bijective linear transformation $h : X \to X'$ satisfying

$$f'(h(x), h(y)) = f(x, y)$$

for all $x, y \in X$. In the special case $(X, f) = (X', f')$, the map h is called an **automorphism**. The set of all such maps forms a group $\operatorname{Aut}(X, f)$ under composition, known as the **automorphism group**. We will need the cardinality $|\operatorname{Aut}(X, f)|$ later when defining the Minkowski-Siegel mass constants.

If X is an n-dimensional $\mathbb{R}$-vector space, then (X, f) is isomorphic to $(\mathbb{R}^n, \cdot)$, that is, Euclidean n-space equipped with the standard dot product [1, 2]. If X is a free $\mathbb{Z}$-module of rank n, then for $n \leq 7$, (X, f) is isomorphic to $(\mathbb{Z}^n, \cdot)$. What happens for $n \geq 8$? A partial answer to this question will occupy us for the remainder of this essay [3, 4, 5].

An inner product module (X, f) over $\mathbb{Z}$ is said to be **even** if $f(x, x) \equiv 0 \bmod 2$ for all $x \in X$. Otherwise it is said to be **odd**. The phrases **Type II** and **Type I** (for even and odd, respectively) are also often used.

There is a more geometric approach to this subject. A **lattice** in $\mathbb{R}^n$ is a subset $\Lambda \subseteq \mathbb{R}^n$ such that, for some basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$, we have

$$\Lambda = \left\{ \sum_{j=1}^{n} i_j e_j : i_j \in \mathbb{Z},\ 1 \leq j \leq n \right\}.$$

The **volume** of Λ is the Lebesgue measure of the fundamental parallelepiped

$$\left\{ \sum_{j=1}^{n} r_j e_j : r_j \in \mathbb{R},\ 0 \leq r_j \leq 1,\ 1 \leq j \leq n \right\}$$

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or, equivalently, the absolute value of the determinant of the matrix whose rows are the vectors e_1, e_2, …, e_n. The lattice Λ is **unimodular** or **self-dual** if the dot product $e_k \cdot e_l \in \mathbb{Z}$ for all $1 \leq k, l \leq n$ and if the volume of Λ is 1. It can be proved that the unimodular lattices in $\mathbb{R}^n$ are "representations" of the free inner product $\mathbb{Z}$-modules of rank n. All properties of one language carry over to the other. For example, a unimodular lattice Λ is even if $v \cdot v \equiv 0 \bmod 2$ for all $v \in \Lambda$; otherwise it is odd [3, 4, 5].

We merely mention that this subject is closely connected with the construction of dense sphere packings in $\mathbb{R}^n$ [6].

0.1. Classification of Inner Product Modules. Classifying pairs (X, f) up to isomorphism, where X is a free $\mathbb{Z}$-module of rank n and f is an inner product, becomes interesting starting at $n = 8$. There is a unique odd module when $n = 8$, namely $(\mathbb{Z}^8, \cdot)$. There is also a unique even module $\mathbb{E}_8$ when $n = 8$; it is easiest to describe $\mathbb{E}_8$ as a certain unimodular lattice in $\mathbb{R}^8$. Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard orthonormal basis of $\mathbb{R}^8$ and define the following to be the basis for $\mathbb{E}_8$:

$$\begin{array}{cccc} 2e_1, & e_2 - e_1, & e_3 - e_2, & e_4 - e_3, \\ e_5 - e_4, & e_6 - e_5, & e_7 - e_6, & \frac{1}{2}(e_1 + e_2 + \cdots + e_8). \end{array}$$

In words, $\mathbb{E}_8$ consists of all points in $\mathbb{R}^8$ whose coordinates are either all integers or all halves of odd integers, and sum to an even integer. We emphasize that $\mathbb{E}_8 \approx \mathbb{Z}^8$ as modules, but $\mathbb{E}_8 \not\approx \mathbb{Z}^8$ as inner product modules [5, 7, 8].

Table 1. *Number of free inner product $\mathbb{Z}$-modules of rank n (Type I and Type II)*

n	a_n	b_n	n	a_n	b_n	n	a_n	b_n
8	1	1	14	4		20	28	
9	2		15	5		21	40	
10	2		16	6	2	22	68	
11	2		17	9		23	117	
12	3		18	13		24	273	24
13	3		19	16		25	665	

Table 1 gives the number a_n of odd unimodular lattices and the number b_n of even unimodular lattices, where $8 \leq n \leq 25$ [7, 9]. For $9 \leq n \leq 11$, the only odd unimodular lattices are $\mathbb{Z}^n$ and $\mathbb{E}_8 \oplus \mathbb{Z}^{n-8}$. When $n = 12$, a third odd lattice $\mathbb{D}_{12}^{+}$ appears. Even unimodular lattices exist if and only if $n \equiv 0 \bmod 8$. When $n = 16$, the only even lattices are $\mathbb{E}_8 \oplus \mathbb{E}_8$ and another new case $\mathbb{D}_{16}^{+}$. The famous Leech lattice $\mathbb{L}$ corresponds to $n = 24$ and is the unique even case with the property that $v \cdot v \geq 4$ for every nonzero $v \in \mathbb{L}$. It is known [10] that $a_{26} \geq 2307$, $a_{27} \geq 14179$, $a_{28} \geq 327972$

and $b_{32} \geq 1162109024 > 10^9$; no one expects a complete classification of even lattices for $n = 32$ to be achieved in the near future.

Against such difficult enumerations, it is surprising that exact formulas, valid for all n, involving the reciprocal sum of automorphism group orders should exist. Let $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, ... denote the Bernoulli numbers and $E_0 = 1$, $E_1 = 0$, $E_2 = -1$, $E_3 = 0$, $E_4 = 5$, ... denote the Euler numbers. The following sum is taken over all nonisomorphic odd unimodular lattices in $\mathbb{R}^n$ [11, 12]:

$$\begin{aligned} M_n &= \sum_{\Lambda} \frac{1}{|\operatorname{Aut}(\Lambda)|} \\ &= \begin{cases} \dfrac{1}{2} & \text{if } n = 1, \\ \dfrac{(1-2^{-k})(1+2^{1-k})}{k! \cdot 2} \left| B_k \cdot B_2 B_4 \cdots B_{2k-2} \right| & \text{if } n = 2k \equiv 0 \bmod 8, \\ \dfrac{2^k+1}{k! \cdot 2^{2k+1}} \left| B_2 B_4 \cdots B_{2k} \right| & \text{if } 1 < n = 2k+1 \equiv \pm 1 \bmod 8, \\ \dfrac{1}{(k-1)! \cdot 2^{2k+1}} \left| E_{k-1} \cdot B_2 B_4 \cdots B_{2k-2} \right| & \text{if } n = 2k \equiv \pm 2 \bmod 8, \\ \dfrac{2^k-1}{k! \cdot 2^{2k+1}} \left| B_2 B_4 \cdots B_{2k} \right| & \text{if } n = 2k+1 \equiv \pm 3 \bmod 8, \\ \dfrac{(1-2^{-k})(1-2^{1-k})}{k! \cdot 2} \left| B_k \cdot B_2 B_4 \cdots B_{2k-2} \right| & \text{if } n = 2k \equiv 4 \bmod 8. \end{cases} \end{aligned}$$

In particular, $M_n = 1/(n!\,2^n)$ for $1 \leq n \leq 8$. Milnor & Husemoller [3] provided a corresponding asymptotic formula:

$$M_n \sim C \cdot \left(\frac{n}{2\pi e \sqrt{e}} \right)^{n^2/4} \left(\frac{8\pi e}{n} \right)^{n/4} \left(\frac{1}{n} \right)^{1/24} = C \cdot F(n)$$

as $n \to \infty$, where $C \approx 0.705$, but no precise expression for C was given. We will return to this issue momentarily. For nonisomorphic even unimodular lattices in $\mathbb{R}^n$, the analogous sum is [4, 11, 12]

$$N_n = \sum_{\Lambda} \frac{1}{|\operatorname{Aut}(\Lambda)|} = \frac{|B_k|}{2k} \prod_{l=1}^{k-1} \frac{|B_{2l}|}{4l}$$

if $n = 2k \equiv 0 \bmod 8$, with asymptotics

$$N_n \sim D \cdot \left(\frac{n}{2\pi e \sqrt{e}} \right)^{n^2/4} \left(\frac{\pi e}{2n} \right)^{n/4} \left(\frac{1}{n} \right)^{1/24}.$$

Such **mass formulas** are useful in verifying that a candidate listing of isomorphism classes of unimodular lattices, for a prescribed genus, is correct. See Tables 2 and 3.

Table 2. *Type I Minkowski-Siegel mass constants* M_n

n	Exact	Decimal
8	$\frac{1}{10321920}$	$9.688... \times 10^{-8}$
9	$\frac{17}{2786918400}$	$6.099... \times 10^{-9}$
10	$\frac{1}{2229534720}$	$4.485... \times 10^{-10}$
$\vdots$		
16	$\frac{505121}{12340763622899712000}$	$4.093... \times 10^{-14}$
17	$\frac{642332179}{1888136834303655936000}$	$3.401... \times 10^{-14}$
18	$\frac{692319119}{15105094674429247488000}$	$4.583... \times 10^{-14}$
$\vdots$		
24	$\frac{70187670795628001881586 2361}{21079028626784998219069784064000000}$	$3.329... \times 10^{-8}$
25	$\frac{8471505948030465162361227284 2147}{30465396080006318014267329085440000000}$	$2.780... \times 10^{-6}$
26	$\frac{1461633563589438887618847268 4851927}{31871491283698917307233513504768000000}$	$4.586... \times 10^{-4}$
27	$\frac{18943527517721468674304869954629232 65007}{1242988160064257774982107026685952000000 0}$	$1.524... \times 10^{-1}$
28	$\frac{1034506037742769404303788948222302395020 3227}{99439052805140621998568562134876160000000}$	$1.040... \times 10^{2}$
29	$\frac{42850098239595906821156287393561695866872 20752159}{28837325313490780379584883019114086400000000}$	$1.485... \times 10^{5}$

Table 3. *Type II Minkowski-Siegel mass constants* N_n

n	Exact	Decimal
8	$\frac{1}{696729600}$	$1.435... \times 10^{-9}$
16	$\frac{691}{277667181515243520000}$	$2.488... \times 10^{-18}$
24	$\frac{1027637932586061520960267}{129477933340026851560636148613120000000}$	$7.936... \times 10^{-15}$
32	$\frac{4890529010450384254108570593011950899382291953107314413193123}{121325280941552041649762780685623131486814208000000000}$	$4.030... \times 10^{7}$

Although M_n and N_n are initially very small and are decreasing, they eventually reverse direction and increase dramatically. The asymptotics for M_n are similar to the asymptotics for the product of even-subscripted Bernoulli numbers:

$$\prod_{j=1}^{n} |B_{2j}| \sim C \cdot n! \cdot 2^{n+1} \cdot F(2n+1).$$

It turns out that the constants C and D can be written as [13]

$$C = 2^{-5/4} e^{1/24} A^{-1/2} Z = 0.7048648734...,$$

$$D = 4C = 2.8194594938... = 2^{1/24} \cdot 2.7391949550...$$

where $A = \exp(\frac{1}{12} - \zeta'(-1)) = 1.2824271291...$ is the Glaisher-Kinkelin constant [14] and

$$Z = \prod_{i=1}^{\infty} \zeta(2i) = 1.8210174514...$$

bears resemblance to certain constants arising when enumerating abelian groups [15].

0.2. Products and Sums of Factorials. While determining C and D, Kellner [13] examined the product of factorials

$$\prod_{\nu=1}^{n}(k\,\nu)! \sim W_k \left(\frac{k\,n}{e\sqrt{e}}\right)^{\frac{k\,n^2}{2}} \left(\frac{k\,n}{e}\right)^{\frac{k\,n}{2}} \left(\frac{2\pi k\,n}{e}\right)^{\frac{n}{2}} n^{\frac{1}{4}+\frac{k}{12}+\frac{1}{12k}}$$

and computed the constants $F_k = (2\pi)^{-1/4}A^{-k}W_k$ to be

$$F_k = k^{\frac{5}{12k}}(2\pi)^{\frac{k}{4}-\frac{1}{2}+\frac{1}{2k}}e^{\frac{1}{12k}}A^{-k-\frac{1}{k}}\prod_{m=2}^{k-1}\Gamma(\tfrac{m}{k})^{-\frac{m-1}{k}}$$

for each positive integer k. In particular, we have

$$F_1 = (2\pi)^{1/4}e^{1/12}A^{-2} = 1.0463350667...,$$

$$F_2 = 2^{5/24}(2\pi)^{1/4}e^{1/24}A^{-5/2} = 1.0239374116...,$$

$$F_3 = 3^{5/36}(2\pi)^{5/12}e^{1/36}A^{-10/3}\Gamma(2/3)^{-1/3} = 1.0160405370...,$$

$$F_4 = 2^{1/3}(2\pi)^{1/2}e^{1/48}A^{-17/4}\Gamma(3/4)^{-1/2} = 1.0120458980....$$

The case $k = 1$ corresponds to the asymptotics of the well-known Barnes G-function [14]. As k grows without bound, we also have

$$\lim_{k\to\infty} F_k = 1, \qquad \lim_{k\to\infty} F_k^k = e^{\gamma/12}$$

where γ is the Euler-Mascheroni constant, and

$$\lim_{l\to\infty} l^{-\gamma/12}\prod_{k=1}^{l} F_k = 1.0246068826....$$

An exact evaluation of the final limit remains open. By way of contrast, the sum of factorials

$$\sum_{\nu=1}^{n}(k\,\nu)! \sim (k\,n)! \sim (2\pi k\,n)^{1/2}\left(\frac{k\,n}{e}\right)^{k\,n}$$

does not involve any new constants.

REFERENCES

[1] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1965, pp. 242, 248; MR0188387 (32 #5826).

[2] A. E. Taylor, *General Theory of Functions and Integration*, Blaisdell, 1965, pp. 154, 161–162; MR0178100 (31 #2358).

[3] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer-Verlag, 1973, pp. 1–3, 15–18, 26–29, 49–51; MR0506372 (58 #22129).

[4] J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, 1973, pp. 48–55; MR0344216 (49 #8956).

[5] W. Ebeling, *Lattices and Codes*, 2nd ed., Friedr. Vieweg & Sohn, 2002, pp. 1–5; MR1280458 (95c:11084) and MR1938666 (2003i:11093).

[6] S. R. Finch, Hermite's constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 506–508.

[7] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, 1988, pp. 41-55, 408–413; MR0920369 (89a:11067), MR1194619 (93h:11069) and MR1662447 (2000b:11077).

[8] G. Nebe and N. J. A. Sloane, Unimodular Lattices, Together With A Table of the Best Such Lattices, available online at http://www.research.att.com/~njas/lattices/unimodular.html.

[9] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A054909 and A054911.

[10] O. D. King, A mass formula for unimodular lattices with no roots, *Math. Comp.* 72 (2003) 839–863; math.NT/0012231; MR1954971 (2003m:11101).

[11] J. H. Conway and N. J. A. Sloane, The unimodular lattices of dimension up to 23 and the Minkowski-Siegel mass constants, *European J. Combin.* 3 (1982) 219–231; MR0679207 (84b:10045).

[12] V. E. Voskresenskii, Calculation of local volumes in the Siegel-Tamagawa formula (in Russian), *Mat. Sbornik* 180 (1989) 443–455, 559; Engl. transl. in *Math. USSR Sbornik* 66 (1990) 447–460; MR0997894 (90h:11055).

[13] B. Kellner, On asymptotic constants related to products of Bernoulli numbers and factorials, *Integers* 9 (2009) 83–106; math.NT/0604505.

[14] S. R. Finch, Glaisher-Kinkelin constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 135–145.

[15] S. R. Finch, Abelian group enumeration constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 273–276.