

Mertens' Formula

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A generalization of Mertens' famous formula recently appeared [1]:

$$\lim_{x \rightarrow \infty} \ln(x)^{1/\varphi(k)} \prod_{\substack{p \leq x, \\ p \equiv \ell \pmod{k}}} \left(1 - \frac{1}{p}\right) = \left[e^{-\gamma} \prod_p \left(1 - \frac{1}{p}\right)^{\alpha(p;k,\ell)} \right]^{1/\varphi(k)}$$

where φ is the Euler totient function, γ is the Euler-Mascheroni constant [2], and $\alpha(p; k, \ell)$ is equal to $\varphi(k) - 1$ if $p \equiv \ell \pmod{k}$ and is -1 otherwise. This constitutes a vast simplification of earlier such formulas [3, 4]. Computing the constant $e^{-\gamma} \Lambda_{k,\ell} = (0.5614594835\dots) \Lambda_{k,\ell}$ inside the square brackets, as well as the related limit [5]:

$$M_{k,\ell} = \lim_{x \rightarrow \infty} \left(\sum_{\substack{p \leq x \\ p \equiv \ell \pmod{k}}} \frac{1}{p} - \frac{1}{\varphi(k)} \ln(\ln(x)) \right)$$

will occupy us for the remainder of this essay.

Let $\zeta(s)$ denote the Riemann zeta function and

$$P_{k,\ell}(s) = \sum_{p \equiv \ell \pmod{k}} \frac{1}{p^s}$$

denote the $(k, \ell)^{\text{th}}$ **prime zeta function** for $\text{Re}(s) > 1$. Clearly $\Lambda_{1,0} = 1$; to efficiently compute $M_{1,0}$, we utilize the series [6, 7, 8]

$$P_{1,0}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln(\zeta(ns)).$$

The numerical evaluation of other $P_{k,\ell}(s)$ will be discussed momentarily. For now, note that

$$\frac{-\gamma + \ln(\Lambda_{k,\ell})}{\varphi(k)} = \lim_{x \rightarrow \infty} \left(\sum_{\substack{p \leq x \\ p \equiv \ell \pmod{k}}} \ln\left(1 - \frac{1}{p}\right) + \frac{1}{\varphi(k)} \ln(\ln(x)) \right);$$

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hence

$$\begin{aligned}
M_{k,\ell} + \frac{\ln(\Lambda_{k,\ell}) - \gamma}{\varphi(k)} &= \sum_{p \equiv \ell \pmod k} \left(\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) \\
&= - \sum_{p \equiv \ell \pmod k} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots \right) \\
&= - \sum_{n=2}^{\infty} \frac{P_{k,\ell}(n)}{n};
\end{aligned}$$

hence

$$M_{1,0} = \gamma - \sum_{n=2}^{\infty} \frac{P_{1,0}(n)}{n} = 0.2614972128\dots$$

In the following two sections, we will discuss the cases $k = 3$ and $k = 4$. In both cases, $\varphi(k) = 2$, which implies that

$$\begin{aligned}
\Lambda_{k,1} &= \frac{k}{2} \prod_{p \equiv 1 \pmod k} \left(1 - \frac{1}{p} \right) \cdot \prod_{p \equiv -1 \pmod k} \left(1 - \frac{1}{p} \right)^{-1} \\
&= \frac{k}{2} \frac{1}{L_{-k}(1)} \prod_{p \equiv -1 \pmod k} \left(1 + \frac{1}{p} \right)^{-1} \cdot \prod_{p \equiv -1 \pmod k} \left(1 - \frac{1}{p} \right)^{-1} \\
&= \frac{k}{2} \frac{1}{L_{-k}(1)} \prod_{p \equiv -1 \pmod k} \left(1 - \frac{1}{p^2} \right)^{-1}
\end{aligned}$$

where L_{-k} is Dirichlet's L-series associated to $(-k/\cdot)$. The infinite product can be evaluated via the $(k, -1)^{\text{th}}$ prime zeta function since

$$\begin{aligned}
\ln \left(\prod_{p \equiv \ell \pmod k} \left(1 - \frac{1}{p^2} \right) \right) &= \sum_{p \equiv \ell \pmod k} \left(\ln \left(1 + \frac{1}{p} \right) + \ln \left(1 - \frac{1}{p} \right) \right) \\
&= - \sum_{p \equiv \ell \pmod k} \left(\frac{1}{p^2} + \frac{1}{2p^4} + \frac{1}{3p^6} + \cdots \right) \\
&= - \sum_{n=1}^{\infty} \frac{P_{k,\ell}(2n)}{n}.
\end{aligned}$$

Thus we first compute $\Lambda_{3,1}$ and $\Lambda_{4,1}$, and then $M_{3,1}$ and $M_{4,1}$.

Let χ_0 denote the principal character mod k and χ_1 denote the nonprincipal character mod k (χ_1 is unique since $k = 3$ or $k = 4$). In order to evaluate $P_{k,1}(s)$ and $P_{k,-1}(s)$, the associated Dirichlet L-series:

$$L_{\chi_j}(s) = \sum_{n=1}^{\infty} \frac{\chi_j(n)}{n^s} = \frac{1}{k^s} \left(\chi_j(1) \zeta \left(s, \frac{1}{k} \right) + \chi_j(-1) \zeta \left(s, 1 - \frac{1}{k} \right) \right), \quad j = 0, 1$$

are required, where $\zeta(s, a)$ is the Hurwitz zeta-function. It can be shown that [9]

$$P_{k,-1}(s) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} \ln \left(\frac{L_{\chi_0}((2n+1)s)}{L_{\chi_1}((2n+1)s)} \right),$$

$$P_{k,1}(s) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} \ln \left(\frac{L_{\chi_0}((2n+1)s)L_{\chi_1}((2n+1)s)}{L_{\chi_0}((4n+2)s)} \right).$$

We will additionally exhibit $\prod_{p \equiv 1 \pmod k} (1 - p^{-2})$ and $\prod_{p \equiv -1 \pmod k} (1 - p^{-2})$, since these are also easily available.

0.1. Residue Classes Mod 3. The two characters modulo 3 are

$$\chi_0(n)|_{n=1,2,3} = \{1, 1, 0\}, \quad \chi_1(n)|_{n=1,2,3} = \{1, -1, 0\};$$

thus $L_{\chi_0}(s) = \zeta(s)(1 - 1/3^s)$ and $L_{\chi_1}(s) = L_{-3}(s)$. We have [10]

$$\prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^2}\right) = 0.9671040753\dots = \frac{9\sqrt{3}}{8} (0.7044984335\dots)^2 = \frac{27\sqrt{3}}{2\pi^2} K_3^2,$$

$$\prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right) = 0.7071813747\dots = \frac{9\sqrt{3}}{2} (0.3012165544\dots)^2 = \frac{\sqrt{3}}{6} \frac{1}{K_3^2}$$

where $K_3 = 0.6389094054\dots$ is the Landau-Ramanujan constant for counting integers of the form $a^2 + 3b^2$ [11]. Also $\Lambda_{3,1} = 27K_3^2/\pi$ and therefore

$$\lim_{x \rightarrow \infty} \sqrt{\ln(x)} \prod_{\substack{p \leq x \\ p \equiv 1 \pmod 3}} \left(1 - \frac{1}{p}\right) = 3\sqrt{\frac{3}{\pi}} e^{-\gamma/2} K_3 = 1.4034774468\dots,$$

$$\lim_{x \rightarrow \infty} \sqrt{\ln(x)} \prod_{\substack{p \leq x \\ p \equiv 2 \pmod 3}} \left(1 - \frac{1}{p}\right) = \frac{1}{2} \sqrt{\frac{\pi}{3}} e^{-\gamma/2} \frac{1}{K_3} = 0.6000732161\dots,$$

$$M_{3,1} = \frac{\gamma}{2} - \ln \left(3\sqrt{\frac{3}{\pi}} K_3 \right) + \sum_{p \equiv 1 \pmod 3} \left[\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] = -0.3568904795\dots,$$

$$M_{3,2} = \frac{\gamma}{2} - \ln \left(\frac{1}{2} \sqrt{\frac{\pi}{3}} \frac{1}{K_3} \right) + \sum_{p \equiv 2 \pmod 3} \left[\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] = 0.2850543590\dots$$

0.2. Residue Classes Mod 4. An alternative approach is given in [12]. The two characters modulo 4 are

$$\chi_0(n)|_{n=1,2,3,4} = \{1, 0, 1, 0\}, \quad \chi_1(n)|_{n=1,2,3,4} = \{1, 0, -1, 0\};$$

thus $L_{\chi_0}(s) = \zeta(s)(1 - 1/2^s)$ and $L_{\chi_1}(s) = L_{-4}(s)$. We have [10]

$$\prod_{p \equiv 1 \pmod{4}} \left(1 - \frac{1}{p^2}\right) = 0.9468064071\dots = 4(0.4865198883\dots)^2 = \frac{16}{\pi^2} K_1^2,$$

$$\prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p^2}\right) = 0.8561089817\dots = 8(0.3271293669\dots)^2 = \frac{1}{2} \frac{1}{K_1^2}$$

where $K_1 = 0.7642236535\dots$ is the (classical) Landau-Ramanujan constant for counting integers of the form $a^2 + b^2$ [11]. Also $\Lambda_{4,1} = 16K_1^2/\pi$ and therefore

$$\lim_{x \rightarrow \infty} \sqrt{\ln(x)} \prod_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \left(1 - \frac{1}{p}\right) = \frac{4}{\sqrt{\pi}} e^{-\gamma/2} K_1 = 1.2923041571\dots,$$

$$\lim_{x \rightarrow \infty} \sqrt{\ln(x)} \prod_{\substack{p \leq x \\ p \equiv 3 \pmod{4}}} \left(1 - \frac{1}{p}\right) = \frac{\sqrt{\pi}}{2} e^{-\gamma/2} \frac{1}{K_1} = 0.8689277682\dots,$$

$$M_{4,1} = \frac{\gamma}{2} - \ln \left(\frac{4}{\sqrt{\pi}} K_1 \right) + \sum_{p \equiv 1 \pmod{4}} \left[\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] = -0.2867420562\dots,$$

$$M_{4,3} = \frac{\gamma}{2} - \ln \left(\frac{\sqrt{\pi}}{2} \frac{1}{K_1} \right) + \sum_{p \equiv 3 \pmod{4}} \left[\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] = 0.0482392690\dots$$

Some low-precision results are known [13] for the residue classes mod 6 and 8; it would be good to repeat these calculations (using the prime zeta function, as above) to high accuracy.

Addendum Languasco & Zaccagnini [14, 15] have proved new formulas and greatly extended the numerical results in this essay.

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