

## Multiples and Divisors

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Before discussing multiplication, let us speak about addition. The number  $A(k)$  of distinct sums  $i + j \leq k$  such that  $1 \leq i \leq k/2$ ,  $1 \leq j \leq k/2$  is clearly  $2 \lfloor k/2 \rfloor - 1$ . Hence the number  $A(2n)$  of distinct elements in the  $n \times n$  addition table involving  $\{1, 2, \dots, n\}$  satisfies  $\lim_{n \rightarrow \infty} A(2n)/n = 2$ , as expected.

We turn to multiplication. Let  $M(k)$  be the number of distinct products  $ij \leq k$  such that  $1 \leq i \leq \sqrt{k}$ ,  $1 \leq j \leq \sqrt{k}$ . One might expect that the number  $M(n^2)$  of distinct elements in the  $n \times n$  multiplication table to be approximately  $n^2/2$ ; for example,  $M(10^2) = 42$ . In a surprising result, Erdős [1, 2, 3] proved that  $\lim_{n \rightarrow \infty} M(n^2)/n^2 = 0$ . More precisely, we have [4]

$$\lim_{k \rightarrow \infty} \frac{\ln(M(k)/k)}{\ln(\ln(k))} = -\delta$$

where

$$\delta = 1 - \frac{1 + \ln(\ln(2))}{\ln(2)} = 0.0860713320\dots$$

In spite of good estimates for  $M(k)$ , an asymptotic formula for  $M(k)$  as  $k \rightarrow \infty$  remains unknown [5].

Given a positive integer  $n$ , define

$$\rho_1(n) = \max_{\substack{d|n, \\ d \leq \sqrt{n}}} d, \quad \rho_2(n) = \min_{\substack{d|n, \\ d \geq \sqrt{n}}} d;$$

thus  $\rho_1(n)$  and  $\rho_2(n)$  are the two divisors of  $n$  closest to  $\sqrt{n}$ . Let

$$R_1(N) = \sum_{n=1}^N \rho_1(n), \quad R_2(N) = \sum_{n=1}^N \rho_2(n).$$

It is not difficult to prove that

$$\lim_{N \rightarrow \infty} \frac{\ln(N)}{N^2} R_2(N) = \frac{\pi^2}{12}.$$

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An analogous asymptotic expression for  $R_1(N)$  is still open, but Tenenbaum [6, 7, 8] proved that

$$\lim_{N \rightarrow \infty} \frac{\ln(R_1(N)/N^{3/2})}{\ln(\ln(N))} = -\delta$$

where  $\delta$  is exactly as before. It is curious that one limit is so much harder than the other, and that the same constant  $\delta$  appears as with the multiplication table problem.

Erdős conjectured long ago that almost all integers  $n$  have two divisors  $d, d'$  such that  $d < d' \leq 2d$ . By “almost all”, we mean all integers  $n$  in a sequence of asymptotic density 1, abbreviated as “p.p.” Given  $n$ , select divisors  $a_n < b_n$  for which  $b_n/a_n$  is minimal. To prove the conjecture, it is sufficient to show that  $b_n/a_n \rightarrow 1^+$  as  $n \rightarrow \infty$  p.p.; that is,  $\ln(\ln(b_n/a_n)) \rightarrow -\infty$  p.p. Maier & Tenenbaum [9, 10, 11] succeeded in doing this and, further, demonstrated that

$$\lim_{n \rightarrow \infty} \frac{\ln(\ln(b_n/a_n))}{\ln(\ln(n))} = -(\ln(3) - 1) = -0.0986122886\dots \text{ p.p.}$$

Another way of viewing this problem is by counting those integers  $n$  up to  $N$  without such divisors  $d$  and  $d'$ . If  $\varepsilon(N)$  is the number of these exceptional integers, then [4]

$$\lim_{N \rightarrow \infty} \frac{\ln(\varepsilon(N)/N)}{\ln(\ln(\ln(N)))} \leq -\beta$$

where

$$\beta = 1 - \frac{1 + \ln(\ln(3))}{\ln(3)} = 0.0041547514\dots$$

As the inequality suggests, we don't know if this constant is necessarily optimal.

Yet another way of viewing this problem is via the Hooley function

$$\Delta(n) = \max_{x \geq 0} \sum_{\substack{d|n, \\ e^x < d \leq e^{x+1}}} 1,$$

that is, the greatest number of divisors of  $n$  contained in any interval of logarithmic length 1. More interesting constants emerge here, but their optimality is questionable. In fact, it is conjectured [4] that  $\Delta(n)/\ln(\ln(n))$  accumulates not at a single point, but over an entire subinterval  $(u, v) \subseteq (0, \infty)$ . Estimates of  $u$  and  $v$  would be good to see someday.

Ramanujan [12] studied the asymptotics of  $\sum_{n=1}^N 1/d(n)$  as  $N \rightarrow \infty$ , where [13]  $d(n)$  is the number of distinct divisors of  $n$ . See [14] for more details. This is a special case of a result in [4, 15], which is used to prove the following arcsine distributional law for random divisors  $d$  of  $n$ :

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \text{P} \left( \frac{\ln(d)}{\ln(n)} < x \right) = \frac{2}{\pi} \arcsin(\sqrt{x}).$$

Consequently, an integer has (on average) many small divisors and many large divisors.

Sita Ramaiah & Suryanarayana [16] found a corresponding formula for  $\sum_{n=1}^N 1/\sigma(n)$ , where [17]  $\sigma(n)$  is the sum of all divisors of  $n$ . DeKoninck & Ivić [18] had asserted that constants appearing in such a formula would be complicated; they were right! [14] It turns out that the Riemann hypothesis [19] is true if and only if [20, 21]

$$\sigma(n) < e^\gamma n \ln(\ln(n)) \quad \text{for all sufficiently large } n,$$

where  $\gamma$  is the Euler-Mascheroni constant [22].

An integer  $n$  is **highly composite** if  $d(m) < d(n)$  for all  $m < n$ . Let  $Q(N)$  denote the number of highly composite integers  $n \leq N$ . It is known that [11, 23, 24, 25, 26]

$$1.136 \leq \liminf_{N \rightarrow \infty} \frac{\ln(Q(N))}{\ln(N)} \leq 1.44, \quad \limsup_{N \rightarrow \infty} \frac{\ln(Q(N))}{\ln(N)} \leq 1.71,$$

based on Diophantine approximations of the quantity  $\ln(3/2)/\ln(2) = 0.5849625007\dots$ . It is conjectured that the limit exists and

$$\lim_{N \rightarrow \infty} \frac{\ln(Q(N))}{\ln(N)} = \frac{\ln(2) + \ln(3) + \ln(5)}{4 \ln(2)} = 1.2267226489\dots$$

but this appears to be difficult.

Let us return to the constant  $\delta$ , which appears in several other places in the literature [27, 28, 29, 30, 31, 32]. We mention only three. With regard to Erdős' conjecture, Roesler [33] added a further constraint that  $a_n b_n = n$  when minimizing  $b_n/a_n$ ; he proved that

$$\lim_{N \rightarrow \infty} \frac{\ln \left( \frac{1}{N} \sum_{n=1}^N \frac{a_n}{b_n} \right)}{\ln(\ln(N))} = -\delta.$$

Hence the integers are fairly quadratic, in the sense that  $b_n - a_n$  is quite small on average. We wonder what happens to the limiting ratio if  $a_n/b_n$  is replaced in the summation by  $b_n/a_n$ .

An odd prime  $p$  is said to be **symmetric** [34, 35] if there exists an odd prime  $q$  such that  $|p - q| = \gcd(p - 1, q - 1)$ . For example, any twin prime is symmetric. It is known that the reciprocal sum of symmetric primes is finite (like Brun's constant [36]). If the twin prime conjecture is true, then there are infinitely many symmetric primes. Let  $S(n)$  denote the number of symmetric primes  $\leq n$ . It is conjectured that

$$\lim_{n \rightarrow \infty} \frac{\ln(S(n)/n)}{\ln(\ln(n))} = -1 - \delta$$

and a heuristic argument supporting this formula appears in [34].

Finally, let  $T(N)$  denote the number of integers  $n \leq N$  satisfying the inequality  $d(n) \geq \ln(N)$ . Norton [37], responding to a question raised by Steinig, proved that there are positive constants  $\xi < \eta$  with

$$\xi \leq \rho(N) = \frac{T(N)}{N \ln(N)^{-\delta} \ln(\ln(N))^{-1/2}} \leq \eta$$

for all large  $N$ . Balazard, Nicolas, Pomerance & Tenenbaum [38] proved that the ratio  $\rho(N)$  does not tend to a limit as  $N \rightarrow \infty$ , and that

$$\rho(N) \sim f\left(\frac{\ln(\ln(N))}{\ln(2)}\right) \quad \text{as } N \rightarrow \infty$$

where  $f(x)$  is an explicit left-continuous function of period 1 with only countably many jump discontinuities. Deléglise & Nicolas [39] further computed that

$$\xi = \lim_{x \rightarrow 0^+} f(x) = 0.9382786811\dots, \quad \eta = f(0) = 1.1481267734\dots$$

are the best possible asymptotic bounds on  $\rho(N)$ . We have seen such oscillatory functions on numerous occasions elsewhere in number theory and combinatorics [40, 41]. The quantities

$$\chi = \frac{1}{\Gamma(1 + \lambda)} \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^\lambda \left(1 + \frac{\lambda}{p}\right) = 0.3495143728\dots$$

$$\frac{\chi}{1 - \ln(2)} \sqrt{\frac{\ln(2)}{2\pi}} = 0.3783186209\dots = \frac{\xi}{2.4801282017\dots} = \frac{\eta}{3.0348143331\dots}$$

also play an intermediate role [39], where  $\lambda = \ln(2)^{-1}$ .

In a late-breaking development, Ford [42] proved that there exist positive constants  $c < C$  such that

$$c \frac{N}{\ln(N)^\delta \ln(\ln(N))^{3/2}} \leq M(N) \leq C \frac{N}{\ln(N)^\delta \ln(\ln(N))^{3/2}}$$

for large  $N$ , and positive constants  $c' < C'$  such that

$$c' \frac{N^{3/2}}{\ln(N)^\delta \ln(\ln(N))^{3/2}} \leq R_1(N) \leq C' \frac{N^{3/2}}{\ln(N)^\delta \ln(\ln(N))^{3/2}}.$$

for large  $N$ . Thus, for the first time, the true order of magnitude of  $M(N)$  and of  $R_1(N)$  is known. See also [43] for an application to computer science.

**0.1. Addendum.** A famous result is [44, 45, 46, 47, 48]

$$\limsup_{n \rightarrow \infty} \ln(d(n)) \frac{\ln \ln n}{\ln n} = \ln 2$$

but the analogous result for the iterated divisor

$$\begin{aligned} \limsup_{n \rightarrow \infty} \ln(d(d(n))) \frac{\ln \ln n}{\sqrt{\ln n}} &= \left( 8 \sum_{j=1}^{\infty} \ln \left( 1 + \frac{1}{j} \right)^2 \right)^{1/2} \\ &= 2.7959802335\dots \end{aligned}$$

was proved only recently [49].

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