Lyapunov Exponents. IV

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We are interested in the effects of multiplicative noise (continuing our study [1]). Let E_n denote matrix $N(0,1)$ white noise, that is, E_1, E_2, E_3, … is a sequence of independent $m \times m$ matrices and all m^2 entries of E_n, for each n, are independent standard normal variables. Cohen & Newman [2] proved that the recurrence

$$X_n = E_n \, X_{n-1}, \qquad X_0 \neq 0 \text{ arbitrary}$$

gives rise to Lyapunov exponent

$$\tfrac{1}{n} \ln |X_n| \to \tfrac{1}{2}\left(\ln(2) + \psi(\tfrac{m}{2})\right) \qquad \text{almost surely as } n \to \infty$$

where $\psi(x)$ is the digamma function and $\gamma = -\psi(1)$ is the Euler-Mascheroni constant [3]. In particular, for $m = 1$,

$$x_n = \varepsilon_n x_{n-1}$$

has Lyapunov exponent $\lambda = -(\ln(2)+\gamma)/2$ and the following Central Limit Theorem holds:

$$\frac{\ln|x_n| - n\,\lambda}{\pi\sqrt{n/8}} \to N(0,1) \qquad \text{as } n \to \infty;$$

for $m = 2$,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \varepsilon_n & \varepsilon_n' \\ \varepsilon_n'' & \varepsilon_n''' \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

has Lyapunov exponent $\lambda = (\ln(2) - \gamma)/2$ and

$$\frac{\ln\sqrt{x_n^2 + y_n^2} - n\,\lambda}{\pi\sqrt{n/24}} \to N(0,1) \qquad \text{as } n \to \infty.$$

Upon constraining certain entries of E_n, relevant Lyapunov exponent calculations become more complicated. Wright & Trefethen [4] found that $\lambda = \ln(1.0574735537...)$ when

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon_{n+1} & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix},$$

$\lambda = \ln(1.1149200917...)$ when

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \varepsilon_{n+1} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix},$$

and $\lambda = \ln(0.9949018837...)$ when

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon'_{n+1} & \varepsilon_{n+1} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}.$$

Upon replacing standard normal variables ε_n by symmetric Bernoulli variables

$$\mathrm{P}\left(\varepsilon_n = 1\right) = \mathrm{P}\left(\varepsilon_n = -1\right) = 1/2,$$

the three preceding examples no longer possess distinct Lyapunov exponents. Viswanath [5, 6] proved that the three **random Fibonacci sequences** each have $\lambda = v$, where

$$v = \ln(1.1319882487...) = 0.1239755988...$$

was computed via a fractal invariance measure on the Stern-Brocot division of the real line. A high-precision estimate of v, due to Bai [7], was based on the cycle expansion method applied to a corresponding Ruelle dynamical zeta function [8, 9, 10]. It is interesting to compare the "almost-sure growth rate"

$$\tfrac{1}{n}\,\mathrm{E}\left(\ln\left|x_n\right|\right) \to v = \ln(1.1319882487...)$$

against the "average growth rate" [11, 12]

$$\tfrac{1}{n}\ln\left(\mathrm{E}\left|x_n\right|\right) \to \ln(\xi) = \ln(1.2055694304...)$$

where ξ has minimal polynomial $\xi^3 + \xi^2 - \xi - 2$. The latter value is larger due to outlying sequences that occur with very small probability. It is difficult to detect the difference experimentally since [13]

$$\tfrac{1}{n}\ln\left(\mathrm{Var}\left|x_n\right|\right) \to \ln(1 + \sqrt{5})$$

and hence $\sim (1 + \sqrt{5})^n$ datapoints are needed to estimate $\mathrm{E}\left|x_n\right|$ adequately.

Embree & Trefethen [14] examined the more general linear recurrence

$$x_{n+1} = x_n + \beta\,\varepsilon_{n+1} x_{n-1}$$

and determined that the critical threshold β^* (below which solutions decay exponentially almost surely; above which solutions grow exponentially almost surely) is

$\beta^* = 0.70258...$. It also appears that the value $\tilde{\beta}$ corresponding to maximal decay is $\tilde{\beta} = 0.36747...$ with Lyapunov exponent $\ln(0.8951...)$.

Chassaing, Letac & Mora [15] examined a different kind of random Fibonacci sequence:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{cases} \begin{pmatrix} x_{n-1} + y_{n-1} \\ y_{n-1} \end{pmatrix} & \text{with probability } 1/2, \\ \begin{pmatrix} x_{n-1} \\ x_{n-1} + y_{n-1} \end{pmatrix} & \text{with probability } 1/2 \end{cases}$$

which reduces to the study of random products of the two nonnegative matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Bai [16] computed that $\lambda = \ln(1.4861851938...) = 0.3962125642...$. Let $\varphi = (1 + \sqrt{5})/2$ denote the Golden mean [17]. Another variation is the random sequence:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{cases} \begin{pmatrix} x_{n-1} + y_{n-1} \\ x_{n-1} \end{pmatrix} & \text{with probability } \varphi - 1 \approx 0.62, \\ \begin{pmatrix} y_{n-1} \\ x_{n-1} + y_{n-1} \end{pmatrix} & \text{with probability } 2 - \varphi \approx 0.38 \end{cases}$$

with associated nonnegative matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this case, λ turns out to be $2v/(\varphi - 1)$, which constitutes another occurrence of Viswanath's constant [7].

Fix $\alpha > 0$. Chassaing, Letac & Mora [15, 18] proved that

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \varepsilon_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

has Lyapunov exponent

$$\lambda = \frac{K_0(\alpha)}{\alpha\, K_1(\alpha)}$$

where ε_n is distributed according to $\mathrm{Exp}(\alpha/2)$ and K_0, K_1 are modified Bessel functions [19]. If $\alpha = 2$, then $2\lambda = K_0(2)/K_1(2) = 0.8143077587...$. A related ratio $I_1(2)/I_0(2)$ appears in [20]; see also [1].

Lyons [21, 22] studied

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon_n \\ 1 & 1 + \varepsilon_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix},$$

where $\varepsilon_n = 0$ with probability $1/2$ and $\varepsilon_n = \tau$ otherwise. It turns out that $\tau \mapsto \lambda(\tau)$ is a strictly increasing function of $\tau > 0$. An important threshold value $\tau = 0.2688513727...$ is the solution of the equation [16]

$$2\lambda(\tau) = \ln(2)$$

and is connected with the distribution of certain random continued fractions.

Ishii [23, 24] proved that

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & c - \varepsilon_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$$

has Lyapunov exponent

$$\lambda(c) = \operatorname{arccosh}\left(\frac{\sqrt{(2+c)^2+\delta^2}+\sqrt{(2-c)^2+\delta^2}}{4}\right)$$

where ε_n is distributed according to Cauchy(δ). If instead ε_n follows a Unif$(-\sqrt{3}\sigma, \sqrt{3}\sigma)$ distribution or a $N(0,\sigma^2)$ distribution, then asymptotic results of Derrida & Gardner [25, 26] apply:

$$\lim_{\sigma\to 0^+} \frac{\lambda(c,\sigma)}{\sigma^{2/3}} = \frac{6^{1/3}\sqrt{\pi}}{2\Gamma(1/6)} = 0.2893082598... \qquad \text{if } c = 2,$$

$$\lim_{\sigma\to 0^+} \frac{\lambda(c,\sigma)}{\sigma^2} = \begin{cases} 1/6 & \text{if } c = 1, \\ \dfrac{\Gamma(3/4)^2}{\Gamma(1/4)^2} = 0.1142366452... = \dfrac{12}{105.0451015308...} & \text{if } c = 0. \end{cases}$$

The constants 0.2893082598... and 0.1142366452... also appear in [27, 28], respectively, but reasons for these connections are unclear.

Fix an odd integer $k \geq 3$. Pincus [29, 30] and Lima & Rahibe [31] examined

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{cases} \begin{pmatrix} \cos(\frac{\pi}{k})x_{n-1} + \sin(\frac{\pi}{k})y_{n-1} \\ -\sin(\frac{\pi}{k})x_{n-1} + \cos(\frac{\pi}{k})y_{n-1} \end{pmatrix} & \text{with probability } 1-\eta, \\ \begin{pmatrix} x_{n-1} \\ 0 \end{pmatrix} & \text{with probability } \eta \end{cases}$$

and proved that

$$\lambda(k) = \frac{\eta^2}{1-(1-\eta)^{2k}} \sum_{j=1}^{2k-1} (1-\eta)^j \ln\left|\cos\left(\frac{j\,\pi}{k}\right)\right|.$$

The identical expression emerges if we replace the definition of the latter portion by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \ell\, x_{n-1} \\ (1/\ell) y_{n-1} \end{pmatrix} \qquad \text{with probability } \eta$$

for a fixed integer $\ell \geq 2$, and compute the asymptotic difference between $\lambda(k, \ell)$ and $\eta \ln(\ell)$ in the limit as $\ell \to \infty$. A precise numerical estimate of $\lambda(3, 2) = 0.1794...$, however, is evidently open [16].

Ben-Naim & Krapivsky [32] studied two variations of random Fibonacci sequences:

$$x_n = \begin{cases} x_{n-1} + x_{n-2} & \text{with probability } 1 - \eta \\ x_{n-1} + x_{n-3} & \text{with probability } \eta \end{cases}, \qquad x_0 = 0, \qquad x_1 = x_2 = 1;$$

$$x_n = \begin{cases} x_{n-1} + x_{n-2} & \text{with probability } 1 - \eta \\ 2x_{n-1} & \text{with probability } \eta \end{cases}, \qquad x_1 = x_2 = 1$$

and determined that

$$\lim_{\eta \to 0^+} \lambda(\eta) = \ln(\varphi)$$

for both cases. Second-order asymptotic terms differ, however:

$$\lim_{\eta \to 0^+} \frac{\lambda(\eta) - \ln(\varphi)}{\eta} = \begin{cases} \ln\left(\dfrac{2\varphi}{\varphi + 2}\right) & \text{for case 1,} \\ \ln\left(\dfrac{2\varphi + 1}{\varphi + 2}\right) & \text{for case 2} \end{cases}$$

and a third-order term is possible for the latter.

Consider the random geometric sequence [33]

$$x_n = 2x_p, \qquad x_0 = 1, \qquad p \in \{0, 1, \ldots, n-1\}$$

where each of the n possible indices is given equal weight. The sequence is not necessarily increasing, but enjoys average growth $n + 1$ and almost-sure growth

$$2^{\gamma} n^{\ln(2)} = (1.4919670404...) \exp\left(\ln(2) \ln(n)\right).$$

Consider instead two additional random Fibonacci models [34, 35]:

$$x_n = x_{n-1} + x_q, \qquad x_0 = 1, \qquad q \in \{0, 1, \ldots, n-1\};$$

$$x_n = x_p + x_q, \qquad x_0 = 1, \qquad p, q \in \{0, 1, \ldots, n-1\}.$$

Model 1 enjoys average growth

$$\frac{1}{2\sqrt{e\,\pi}} n^{-1/4} \exp(2\sqrt{n})$$

and almost-sure growth

$$C\exp\left((1.889...)\sqrt{n}\right)$$

where $C>0$ is unknown. Model 2 is not necessarily increasing but enjoys average growth $n+1$; unlike the random geometric sequence, it seems not to display almost-sure behavior of any kind.

Kenyon & Peres [36] studied random products associated with two sets of matrices:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}.$$

The three matrices in the first set are equiprobable, with Lyapunov exponent $\ln(2)/3 = 0.2310490601....$ The four matrices in the second set are likewise equiprobable, with Lyapunov exponent [37]

$$\frac{1}{6}\ln\left(\frac{2}{3}\right)+\sum_{i=0}^{\infty}4^{-i-1}\ln\left(\frac{(3\cdot 2^i)!}{(2^{i+1})!}\right)=0.7974350484....$$

We wonder whether exp(0.7974350484...) is transcendental. Moshe [38] studied random products associated with two equiprobable 3×3 matrices:

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ -3 & -6 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 8 \\ -2 & -1 & -4 \\ 3 & 1 & 4 \end{pmatrix}$$

and computed Lyapunov exponent

$$\frac{1}{16}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^{j+k}}\ln\left|3\cdot 2^{3j}-2(-1)^j-\tfrac{22}{9}2^{3j+k}+\tfrac{22}{9}(-1)^j2^k\right|=0.5897925607....$$

Many more similar examples are found in [39, 40].

Up to now, the random mechanisms underlying sequences have been very simple. Here is a more complicated but well-known example [41, 42]:

$$x_{n+1}=a_nx_n+x_{n-1}, \qquad x_0=0, \qquad x_1=1$$

where the cofficients a_n are obtained by selecting a random $\theta\in[0,1]$ and computing its continued fraction digits:

$$\theta=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots.$$

For instance, if $\theta = \pi - 3$, then

$$\{a_1, a_2, a_3, a_4\} = \{7, 15, 1, 292\}, \qquad \{x_2, x_3, x_4, x_5\} = \{7, 106, 113, 33102\};$$

note that x_n is simply the denominator of the n^{th} partial convergent to θ. Lévy [43] proved that this recurrence gives rise to Lyapunov exponent

$$\frac{\pi^2}{12\ln(2)} = 1.1865691104....$$

Another example involves the recurrence [44]

$$x_{n+1} = 2^{b_n} x_n + 2^{b_{n-1}} x_{n-1}, \qquad x_0 = 0, \qquad x_1 = 1$$

where the cofficients b_n are obtained via

$$\theta = \frac{2^{-b_1}|}{|1} + \frac{2^{-b_2}|}{|1} + \frac{2^{-b_3}|}{|1} + \cdots.$$

The corresponding Lyapunov exponent is

$$\frac{1}{\ln(4/3)}\left(\frac{\pi^2}{12} + \operatorname{Li}_2\left(-\frac{1}{2}\right)\right) = 1.3002298798...$$

where $\operatorname{Li}_2(y)$ is the dilogarithm function [45]. (This constant also appears in [46] without explanation.) Generalization to base $k \geq 2$ is possible, as well as formulation for Khintchine-type and Lochs-type constants in this broad setting.

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