

## Lyapunov Exponents

STEVEN FINCH

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We are interested in iterates of the **logistic map**  $T : [0, 1] \rightarrow [0, 1]$  defined by

$$T(x) = ax(1-x)$$

where  $0 \leq a \leq 4$  is constant. Actually, only the values  $a = 4$  and

$$a = \frac{2}{3} \left( \left( (19 + 3\sqrt{33})^{1/3} + 4 \left( (19 + 3\sqrt{33})^{-1/3} + 1 \right) \right) = 3.6785735104\dots \right.$$

will be examined (the latter has minimal polynomial  $a^3 - 2a^2 - 4a - 8$ ). Both correspond to chaotic maps for which invariant probability densities  $f(x)$  provably exist. An important feature of chaos is sensitivity to initial conditions. The **Lyapunov exponent** for each map quantifies the exponential rate at which two initially close points  $x, y$  separate [1]:

$$|T(x) - T(y)| \approx |T'(x)| \cdot |x - y|$$

after the first iteration,

$$|T^n(x) - T^n(y)| \approx \prod_{0 \leq j < n} |T'(T^j x)| \cdot |x - y|$$

after the  $n^{\text{th}}$  iteration, and hence

$$\frac{1}{n} \ln |T^n(x) - T^n(y)| \approx \frac{1}{n} \sum_{0 \leq j < n} \ln |T'(T^j x)|.$$

For  $X$  distributed according to  $f$ , let us write

$$\hat{\mu}_n(X) = \frac{1}{n} \sum_{0 \leq j < n} T^j X, \quad \hat{\lambda}_n(X) = \frac{1}{n} \sum_{0 \leq j < n} \ln |T'(T^j X)|$$

which converge as  $n \rightarrow \infty$  almost surely, by ergodicity, to

$$\mathbb{E}(X) = \int_0^1 x f(x) dx, \quad \mathbb{E} |\ln(T' X)| = \int_0^1 \ln |T'(x)| f(x) dx.$$

Our study will encompass not only means, but also variances and autocorrelations of arbitrary time lag. A complete solution is possible for  $a = 4$ ; only partial results exist for  $a = 3.678\dots$ . The approach we take is similar to [2].

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**0.1. Ulam-von Neumann Map.** When  $a = 4$ , the invariant density has a closed-form expression [3]:

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

and thus

$$\mathbb{E}(T^j X) = \frac{1}{2}, \quad \text{Var}(T^j X) = \frac{1}{8}, \quad \text{Cov}(T^j X, T^k X) = 0$$

for all  $j < k$ . Also [4],

$$\mathbb{E} |\ln(T'(T^j X))| = \ln(2), \quad \text{Var} |\ln(T'(T^j X))| = \frac{\pi^2}{12},$$

$$\text{Cov} \left( |\ln(T'(T^j X))|, |\ln(T'(T^k X))| \right) = -\frac{\pi^2}{24} \frac{1}{2^{k-j}}$$

for all  $j < k$ . Clearly

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}(\hat{\mu}_n(X)) &= \frac{1}{2}, \\ \lim_{n \rightarrow \infty} n \text{Var}(\hat{\mu}_n(X)) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{0 \leq j < n, \\ 0 \leq k < n}} \text{Cov}(T^j X, T^k X) = \frac{1}{8} \end{aligned}$$

and the Central Limit Theorem holds:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( 2\sqrt{2n} \left( \hat{\mu}_n(X) - \frac{1}{2} \right) \leq t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp \left( -\frac{u^2}{2} \right) du.$$

By contrast,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}(\hat{\lambda}_n(X)) &= \ln(2), \\ \lim_{n \rightarrow \infty} n^2 \text{Var}(\hat{\lambda}_n(X)) &= \lim_{n \rightarrow \infty} \sum_{\substack{0 \leq j < n, \\ 0 \leq k < n}} \text{Cov} \left( |\ln(T'(T^j X))|, |\ln(T'(T^k X))| \right) \\ &= \lim_{n \rightarrow \infty} \frac{\pi^2}{6} \left( 1 - \frac{1}{2^n} \right) = \frac{\pi^2}{6}. \end{aligned}$$

Estimates  $\hat{\lambda}_n(X)$  of the Lyapunov exponent are anomalously precise [5]: they possess a standard deviation that scales as  $1/n$  rather than  $1/\sqrt{n}$ . In this case, evidence points to a revised Central Limit Theorem of the form [4, 6]:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( n \left( \hat{\lambda}_n(X) - \ln(2) \right) \leq t \right) = \frac{2}{\pi^2} \int_{-\infty}^t \ln \left( \coth \left( \frac{u}{2} \right) \right) du$$

but a rigorous proof seems to be open.

**0.2. Ruelle-Misiurewicz Map.** When  $a = 3.678\dots$ , no closed-form expression for the invariant density is known, even though its existence is certain [7, 8, 9]. A numerical approach is necessary. Let  $y = \frac{1}{2}a - ax$ , then under the change of variables,  $T$  becomes

$$S(y) = y^2 - c$$

where

$$c = \frac{1}{4}a^2 - \frac{1}{2}a = 1.5436890126\dots$$

(with minimal polynomial  $c^3 - 2c^2 + 2c - 2$ ). Now let [10]

$$\theta = \frac{1}{\pi} \arccos\left(\frac{y}{c - c^2}\right);$$

under this second change of variables,  $S^2$  becomes

$$\tau(\theta) = \frac{1}{\pi} \arccos\left(\cos(2\pi\theta) + \kappa \sin(2\pi\theta)^2\right)$$

where

$$\kappa = \frac{c(c^2 - c) - 1}{2} = 0.1477988712\dots$$

The invariant density  $\varphi$  associated with  $\tau : [0, 1] \rightarrow [0, 1]$  satisfies the functional equation [10]

$$\frac{\varphi(\tau^{-1}(\theta))}{|\tau'(\tau^{-1}(\theta))|} + \frac{\varphi(1 - \tau^{-1}(\theta))}{|\tau'(1 - \tau^{-1}(\theta))|} = \varphi(\theta)$$

where

$$\tau^{-1}(\theta) = \frac{1}{2\pi} \arccos\left(\frac{1 - \sqrt{1 + 4\kappa^2 - 4\kappa \cos(\pi\theta)}}{2\kappa}\right),$$

$$\tau'(\theta) = \frac{2 \sin(2\pi\theta) (1 - 2\kappa \cos(2\pi\theta))}{\sqrt{\sin(2\pi\theta)^2 (1 - 2\kappa \cos(2\pi\theta) - \kappa^2 \sin(2\pi\theta)^2)}}.$$

The left-hand side of this equation is a special case of the Frobenius-Perron operator  $P_\tau \varphi(\theta)$ . Starting with an initial guess  $\varphi_0 \equiv 1$ , the uniform density, iterates  $\varphi_{n+1} = P_\tau \varphi_n$  converge to a limiting density  $\varphi$ . Backtracking through the two coordinate transformations, we obtain the desired invariant density  $f$ . It turns out to be supported on the intervals  $[\frac{1}{a}, 1 - \frac{1}{a}]$  and  $[1 - \frac{1}{a}, \frac{a}{4}]$ , which are exchanged by  $T$ , with three vertical asymptotes.

Recall that  $x = \frac{1}{2} - \frac{1}{a}y$  and  $y = (c - c^2) \cos(\pi\theta)$ . For  $X$  distributed according to  $f$ , we compute

$$E(X) = \frac{1}{2} \int_0^1 (x + T(x)) \varphi(\theta) d\theta = 0.6717404535\dots,$$

$$E |\ln(T'X)| = \frac{1}{2} \int_0^1 (\ln |2y| + \ln |2S(y)|) \varphi(\theta) d\theta = 0.3421726886\dots$$

No one evidently has computed higher-order moments of  $X$  and  $\ln(T'X)$ , let alone  $\hat{\mu}_n(X)$  and  $\hat{\lambda}_n(X)$ . Does the Central Limit Theorem need revision here too?

The value 3.678... is the simplest *Misiurewicz point*. For any  $a \leq 3.678\dots$ , the logistic map  $T$  admits no periodic point  $x$  of odd order  $> 1$ , i.e., it has no odd cycles. For any  $a > 3.678\dots$ ,  $T$  has odd cycles [11, 12].

A graph of  $E(X)$ , as a function of  $a$ , appears in [13]; the more familiar graph of  $E |\ln(T'X)|$  appears in [14]. In a sense, such plotting is meaningless, because there always exists finer detail than captured in whatever scale we choose [15].

Jakobson [16, 17] proved that the set  $A = \{a \in [0, 4] : T \text{ has an absolutely continuous invariant density}\}$  has positive measure. Both  $4 \in A$  and  $3.678\dots \in A$ , but the status of values like 3.6, 3.7, 3.8 or 3.9 is unknown. Note: the condition that a density be *absolutely continuous* is important, yet outside our scope of study. What can be said about  $T$  for  $a \notin A$ ? This question was satisfactorily answered only recently [18, 19].

The *metric entropy* of  $T$  can be proved to be equal to the Lyapunov exponent, but the *topological entropy* is altogether a different characterization [20, 21, 22, 23]. For the regular continued fraction transformation  $T_{\text{RCF}}(x) = \{1/x\}$ , the metric entropy is  $\pi^2/(6 \ln(2))$  while the topological entropy is infinite [24]. The limit of  $E(\hat{\lambda}_n(X))$  as  $n \rightarrow \infty$  is  $\pi^2/(6 \ln(2))$ ; the limit of  $n \text{Var}(\hat{\lambda}_n(X))$  as  $n \rightarrow \infty$  is equal to  $4(0.8621470373\dots)$  and, in fact, the Central Limit Theorem holds [2].

One-dimensional maps of the interval have inspired much computation [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. We mention, for example, the maps  $T_\ell : [0, 1] \rightarrow [0, 1]$  defined by

$$T_\ell(x) = 1 - |2x - 1|^\ell$$

for real  $\ell > 1$ . Clearly the case  $\ell = 2$  gives the Ulam-von Neumann map. Each  $T_\ell$  has an absolutely continuous invariant density with metric entropies (Lyapunov exponents) equal to [32, 34]

$$\left\{ \begin{array}{ll} \ln(2) = 0.6931471805\dots & \text{if } \ell = 2, \\ 0.6908569334\dots & \text{if } \ell = 3, \\ 0.6844935750\dots & \text{if } \ell = 4, \\ 0.6756910613\dots & \text{if } \ell = 5. \end{array} \right.$$

As another example, consider the map  $S_0 : [0, 1] \rightarrow [0, 1]$  defined by

$$S_0(x) = \left\{ 2x + \frac{1}{4\pi} \sin(2\pi x) \right\}.$$

The absolutely continuous invariant density of  $S_0$  has entropy equal to 0.6837719602.... It would be good someday to see such high-precision results for the logistic map, given values of  $a$  other than 3.678... and 4.

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