

Continued Fraction Transformation

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We are interested in iterates of the **continued fraction transformation** $T : [0, 1] \rightarrow [0, 1]$ defined by [1]

$$T(x) = \begin{cases} \left\{ \frac{1}{x} \right\} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

where $\{\xi\} = \xi - \lfloor \xi \rfloor$ denotes the fractional part of ξ . For example,

$$\begin{array}{ll} \pi - 3 = 0.141592\dots, & \left\lfloor \frac{1}{\pi-3} \right\rfloor = 7, \\ T(\pi - 3) = 0.062513\dots, & \left\lfloor \frac{1}{T(\pi-3)} \right\rfloor = 15, \\ T^2(\pi - 3) = 0.996594\dots, & \left\lfloor \frac{1}{T^2(\pi-3)} \right\rfloor = 1, \\ T^3(\pi - 3) = 0.003417\dots, & \left\lfloor \frac{1}{T^3(\pi-3)} \right\rfloor = 292, \\ T^4(\pi - 3) = 0.634591\dots, & \left\lfloor \frac{1}{T^4(\pi-3)} \right\rfloor = 1 \end{array}$$

and

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \dots$$

is the regular continued fraction expansion for π . In words, T discards the first “digit” in any expansion, that is,

$$T\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots\right) = \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \dots.$$

What can be said about the moments of $T^j X$ and of $\ln(T^j X)$, where X is a random variable in $[0, 1]$? There are two cases: the first when X follows the uniform distribution, and the second when X follows the **Gauss-Kuzmin distribution**:

$$P(X \leq x) = \frac{\ln(x+1)}{\ln(2)}.$$

We will later study the partial convergents to x , for example,

$$\frac{p_1}{q_1} = \frac{3}{1}, \quad \frac{p_2}{q_2} = \frac{22}{7}, \quad \frac{p_3}{q_3} = \frac{333}{106}, \quad \frac{p_4}{q_4} = \frac{355}{113}, \quad \frac{p_5}{q_5} = \frac{103993}{33102}, \quad \dots$$

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when $x = \pi$. The asymptotic distribution of denominators Q_n , corresponding to uniformly distributed X as $n \rightarrow \infty$, turns out to be related to our earlier work on $\ln(T^j X)$ statistics.

0.1. Uniform Distribution. Let γ denote the Euler-Mascheroni constant [2], ζ denote the Riemann zeta function and Li_k denote the k^{th} polylogarithm function [3]. If X is a random variable following the uniform distribution on $[0, 1]$, then

$$\mathbb{E}(X) = \int_0^1 x \, dx = \frac{1}{2}, \quad \mathbb{E}(X^2) = \int_0^1 x^2 \, dx = \frac{1}{3},$$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{12}$$

and, via the substitution $y = 1/x$,

$$\begin{aligned} \mathbb{E}(TX) &= \int_0^1 \left\{ \frac{1}{x} \right\} dx = \int_1^\infty \frac{\{y\}}{y^2} dy = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{y-n}{y^2} dy \\ &= \sum_{n=1}^{\infty} \left(\ln \left(\frac{n+1}{n} \right) - \frac{1}{n+1} \right) = 1 - \gamma = 0.4227843351\dots \end{aligned}$$

(which is related to de la Vallée Poussin's theorem [2, 4]),

$$\mathbb{E}((TX)^2) = \ln(2\pi) - \gamma - 1,$$

$$\text{Var}(TX) = \ln(2\pi) - \gamma^2 + \gamma - 2 = 0.0819148075\dots = (0.2862076300\dots)^2,$$

$$\mathbb{E}(X \cdot TX) = 1 - \frac{\pi^2}{12},$$

$$\text{Cov}(X, TX) = \mathbb{E}(X \cdot TX) - \mathbb{E}(X) \mathbb{E}(TX) = \frac{1}{12} (6 - \pi^2 + 6\gamma),$$

$$\rho(X, TX) = \frac{\text{Cov}(X, TX)}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(TX)}} = \frac{6 - \pi^2 + 6\gamma}{\sqrt{12} \sqrt{\ln(2\pi) - \gamma^2 + \gamma - 2}} = -0.4098133678\dots$$

where ρ denotes cross-correlation. Likewise,

$$\mathbb{E}(\ln(X)) = -1, \quad \mathbb{E}(\ln(X)^2) = 2, \quad \text{Var}(\ln(X)) = 1,$$

and, via the substitutions $y = 1/x$ and $z = y - n$,

$$\mathbb{E}(\ln(TX)) = \int_0^1 \ln \left\{ \frac{1}{x} \right\} dx = \int_1^\infty \frac{\ln \{y\}}{y^2} dy = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\ln(y-n)}{y^2} dy$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \int_0^1 \frac{\ln(z)}{(z+n)^2} dz = - \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(\frac{n+1}{n}\right) \\
&= - \left(\ln(2) + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k-1} \right) = -1.2577468869\dots
\end{aligned}$$

(this constant appears elsewhere [5, 6]),

$$E(\ln(TX)^2) = -2 \sum_{n=1}^{\infty} \frac{1}{n} \text{Li}_2\left(-\frac{1}{n}\right) = \zeta(2) - 2 \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(k+1) - 1}{k^2},$$

$$\text{Var}(\ln(TX)) = 1.2665694005\dots = (1.1254196552\dots)^2,$$

$$\begin{aligned}
E(\ln(X) \cdot \ln(TX)) &= \sum_{n=1}^{\infty} \frac{1}{n} \left[\ln\left(\frac{n+1}{n}\right) (1 + \ln(n)) - \text{Li}_2\left(\frac{1}{n+1}\right) \right] \\
&= -\zeta(2) + \sum_{k=2}^{\infty} \left[\left(\zeta(2) - \sum_{\ell=1}^{k-1} \frac{1}{\ell^2} \right) (\zeta(k) - 1) - \left(1 + \frac{(-1)^k}{k-1} \right) \zeta'(k) \right],
\end{aligned}$$

$$\rho(\ln(X), \ln(TX)) = -0.2275522084\dots$$

The cumulative distribution for TX can be expressed in terms of the digamma function:

$$F(x) = P(TX \leq x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x} \right) = \gamma + \psi(x+1),$$

and its density in terms of the trigamma function:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} = \psi'(x+1).$$

For example, the median of TX is $F^{-1}(1/2) = 0.3846747346\dots$. The cumulative distribution for T^2X is

$$\begin{aligned}
G(x) &= P(T^2X \leq x) = \sum_{n=1}^{\infty} \left(F\left(\frac{1}{n}\right) - F\left(\frac{1}{n+x}\right) \right) \\
&= \sum_{n=1}^{\infty} \left(\psi\left(\frac{1}{n} + 1\right) - \psi\left(\frac{1}{n+x} + 1\right) \right),
\end{aligned}$$

its density is

$$g(x) = \sum_{n=1}^{\infty} \psi'\left(\frac{1}{n+x} + 1\right) \frac{1}{(n+x)^2},$$

and its median is $G^{-1}(1/2) = 0.42278\dots$. It is certainly inconvenient that $F \neq G$!

0.2. Gauss-Kuzmin Distribution. If X is a random variable following the Gauss-Kuzmin distribution on $[0, 1]$, then

$$\mathbb{E}(X) = \frac{1}{\ln(2)} - 1 = 0.4426950408\dots = \mathbb{E}(TX),$$

$$\mathbb{E}(X^2) = 1 - \frac{1}{2\ln(2)} = \mathbb{E}((TX)^2),$$

$$\text{Var}(X) = \frac{(3/2)\ln(2) - 1}{\ln(2)^2} = 0.0826735803\dots = (0.2875301381\dots)^2 = \text{Var}(TX)$$

by invariance under T , and

$$\mathbb{E}(X \cdot TX) = 1 - \frac{\gamma}{\ln(2)}, \quad \text{Cov}(X, TX) = \frac{(2 - \gamma)\ln(2) - 1}{\ln(2)^2},$$

$$\rho(X, TX) = \frac{(2 - \gamma)\ln(2) - 1}{(3/2)\ln(2) - 1} = -0.3474517057\dots$$

Likewise,

$$\mathbb{E}(\ln(X)) = -\frac{\pi^2}{12\ln(2)} = -1.1865691104\dots = \mathbb{E}(\ln(TX)),$$

$$\mathbb{E}(\ln(X)^2) = \frac{3\zeta(3)}{2\ln(2)} = \mathbb{E}(\ln(TX)^2),$$

$$\begin{aligned} \text{Var}(\ln(X)) &= \frac{216\ln(2)\zeta(3) - \pi^4}{144\ln(2)^2} = 1.1933560457\dots \\ &= (1.0924083695\dots)^2 = \text{Var}(\ln(TX)), \end{aligned}$$

$$\begin{aligned} \mathbb{E}(\ln(X) \cdot \ln(TX)) &= \frac{1}{\ln(2)} \sum_{n=1}^{\infty} \left[\frac{1}{2} \ln\left(\frac{n+1}{n}\right)^2 \ln((n+1)n) + \ln(n) \text{Li}_2\left(-\frac{1}{n}\right) \right. \\ &\quad \left. - \ln(n+1) \text{Li}_2\left(-\frac{1}{n+1}\right) + \ln(n+1) \text{Li}_2\left(\frac{1}{(n+1)^2}\right) \right. \\ &\quad \left. + 2 \text{Li}_3\left(-\frac{1}{n}\right) - 2 \text{Li}_3\left(-\frac{1}{n+1}\right) + \text{Li}_3\left(\frac{1}{(n+1)^2}\right) \right] \\ &= \frac{1}{\ln(2)} \left[-\frac{3\zeta(3)}{2} + \sum_{k=1}^{\infty} \left(\frac{\zeta(2k) - 1}{k^3} - \frac{\zeta'(2k)}{k^2} + \frac{\zeta''(2k)}{2k} \right) \right], \end{aligned}$$

$$\rho(\ln(X), \ln(TX)) = -0.1858801270\dots = r_1.$$

The median of $T^j X$ is $\sqrt{2} - 1 = 0.4142135623\dots$ for every j . We wish to understand the decay rate of $\rho(X, T^j X)$ and $\rho(\ln(X), \ln(T^j X))$ as j increases, but this appears to be a difficult problem.

0.3. Variance of Sample Mean. Let us consider the sample mean

$$\hat{\mu}_n(X) = -\frac{1}{n} \sum_{0 \leq j < n} \ln(T^j X),$$

that is, the average of the time series $\ln(X), \ln(TX), \dots, \ln(T^{n-1}X)$ built from iterates of T evaluated at X . (The negative sign will simplify subsequent formulation.) It can be proved that

$$\lim_{n \rightarrow \infty} \mathbb{E}(\hat{\mu}_n(X)) = \frac{\pi^2}{12 \ln(2)} = 1.1865691104\dots = \mu,$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n \operatorname{Var}(\hat{\mu}_n(X)) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{0 \leq j < n, \\ 0 \leq k < n}} \operatorname{Cov}(\ln(T^j X), \ln(T^k X)) = \sigma^2 \\ &\approx \frac{216 \ln(2) \zeta(3) - \pi^4}{144 \ln(2)^2} \left(1 + \frac{2r_1}{1 - r_1}\right) \approx 0.8 \end{aligned}$$

for a wide variety of initial distributions for X on $[0, 1]$. The latter is a poor numerical estimate (since it presumes that the lag- ℓ correlation r_ℓ is approximately r_1^ℓ , which is not true). It is inspired, in part, by Salamin [7]. A more precise estimate will be given shortly.

0.4. Partial Convergents. The denominator $Q_n(X)$ of the n^{th} partial convergent to X is connected to our exposition via the formula

$$\underbrace{\ln(Q_n(X))}_{A_n} = - \underbrace{\sum_{0 \leq j < n} \ln(T^j X)}_{B_n} + \varepsilon_n$$

where $|\varepsilon_n| < c$ for all n , for some constant c . It is clear that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(A_n)}{n} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(B_n)}{n} = \mu$$

and further known [8] that

$$0 < \lim_{n \rightarrow \infty} \frac{\operatorname{Var}(A_n)}{n} < \infty.$$

We wish to prove that

$$\lim_{n \rightarrow \infty} \frac{\operatorname{Var}(A_n)}{n} = \lim_{n \rightarrow \infty} \frac{\operatorname{Var}(B_n)}{n}.$$

From $B_n = A_n - \varepsilon_n$, deduce that

$$\text{Var}(B_n) = \text{Var}(A_n) - 2 \text{Cov}(A_n, \varepsilon_n) + \text{Var}(\varepsilon_n);$$

hence

$$\begin{aligned} |\text{Var}(A_n) - \text{Var}(B_n)| &\leq 2 |\text{Cov}(A_n, \varepsilon_n)| + \text{Var}(\varepsilon_n) \\ &\leq 2\sqrt{\text{Var}(A_n) \text{Var}(\varepsilon_n)} + \text{Var}(\varepsilon_n) \\ &\leq 2\sqrt{\text{Var}(A_n) \mathbf{E}(\varepsilon_n^2)} + \mathbf{E}(\varepsilon_n^2) \\ &\leq 2c\sqrt{\text{Var}(A_n)} + c^2; \end{aligned}$$

hence

$$\left| \frac{\text{Var}(A_n)}{n} - \frac{\text{Var}(B_n)}{n} \right| \leq 2c\sqrt{\frac{\text{Var}(A_n)}{n^2}} + \frac{c^2}{n} \rightarrow 0$$

as $n \rightarrow \infty$. In particular,

$$\text{Var}(\ln(Q_n(X))) \sim \sigma^2 n$$

and the importance of computing σ^2 (as attempted using iterates of T) becomes evident.

In fact, the existence of σ^2 (in connection with the denominators Q_n) has been known for a long time. Ibragimov [9], Philipp [10, 11, 12] and others [13, 14, 15, 16, 17, 18, 19] proved the following Central Limit Theorem:

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{\frac{1}{n} \ln(Q_n(X)) - \mu}{\frac{\sigma}{\sqrt{n}}} \leq t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp\left(-\frac{u^2}{2}\right) du.$$

No numerical estimate of σ^2 appeared until Flajolet & Vallée [8, 20, 21] computed that

$$\begin{aligned} \sigma^2 &= \lambda_1''(2) - \lambda_1'(2)^2 = 0.8621470373\dots = (0.9285187329\dots)^2 \\ &= \frac{1}{4}(9.0803731646\dots) - \mu^2 = (0.5160624088\dots) \cdot \mu^3, \end{aligned}$$

where $\lambda_1(s)$ is the dominant eigenvalue of a family of linear operators (indexed by s) on a certain infinite-dimensional function space. Lhote [22, 23] proved that σ^2 is polynomial-time computable and obtained higher accuracy. An elementary expression for σ^2 seems to be impossible. The quantities $4\lambda_1''(2)$ or σ^2/μ^3 are often called **Hensley's constant**.

We close with Loch's theorem [1, 24, 25]:

$$\lim_{n \rightarrow \infty} \frac{m(n, x)}{n} = \frac{6 \ln(2) \ln(10)}{\pi^2} = 0.9702701143\dots = (1.0306408341\dots)^{-1} = \alpha$$

for almost all real x , where $m(n, x)$ is the number of partial denominators of x correctly predicted by the first n decimal digits of x . A corresponding Central Limit Theorem was proved by Faivre [26, 27]:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{\frac{m(n, X)}{n} - \alpha}{\frac{\theta}{\sqrt{n}}} \leq t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp \left(-\frac{u^2}{2} \right) du$$

where

$$\begin{aligned} \theta^2 &= \frac{\alpha \sigma^2}{\mu^2} = \frac{864 \ln(2)^3 \ln(10)}{\pi^6} \sigma^2 \\ &= 0.5941388048\dots = (0.7708039990\dots)^2. \end{aligned}$$

For example, the first 10000 decimal digits of π give 9757 partial denominators, consistent with the value of α . A similar empirical confirmation of the value of θ would be good to see.

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