Prophet Inequalities

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Suppose that you view a sequence X_1, X_2, ..., X_n of independent identically distributed nonnegative random variables and that you wish to stop at a value of X as large as possible. As in [1], revisiting earlier values is not permitted. If you are a prophet (meaning that you have complete foresight), then you know $\max\{X_1, \ldots, X_n\}$ beforehand; let M_n denote the average such "insider information" value. If you are a mortal (meaning that you have no choice but to select an X via stopping rules) and if you proceed optimally, then the value V_n obtained satisfies

$$\frac{M_n}{V_n} \leq 1 + \alpha_n$$

for best constants α_n with $0.1 < \alpha_n < 0.6$. Let us now be more precise [2, 3, 4, 5, 6, 7].

Define

$$f_n(w, x) = \frac{n}{n-1} w^{(n-1)/n} + \frac{1}{n-1} x,$$

$$g_{k,n}(x) = \begin{cases} f_n(g_{k-1,n}(x), x) & \text{if } 1 \leq k \leq n, \\ f_n(0, x) & \text{if } k = 0 \end{cases}$$

then α_n is the unique solution of $g_{n-1,n}(x) = 1$, $0 < x < 1$. For example [5, 7],

$$g_{1,2}(x) = \frac{2}{1}\left(\frac{x}{1}\right)^{1/2} + \frac{x}{1},$$

$$g_{2,3}(x) = \frac{3}{2}\left(\frac{3}{2}\left(\frac{x}{2}\right)^{2/3} + \frac{x}{2}\right)^{2/3} + \frac{x}{2},$$

$$g_{3,4}(x) = \frac{4}{3}\left(\frac{4}{3}\left(\frac{4}{3}\left(\frac{x}{3}\right)^{3/4} + \frac{x}{3}\right)^{3/4} + \frac{x}{3}\right)^{3/4} + \frac{x}{3}$$

$$g_{4,5}(x) = \frac{5}{4}\left(\frac{5}{4}\left(\frac{5}{4}\left(\frac{5}{4}\left(\frac{x}{4}\right)^{4/5} + \frac{x}{4}\right)^{4/5} + \frac{x}{4}\right)^{4/5} + \frac{x}{4}\right)^{4/5} + \frac{x}{4}$$

give rise to $\alpha_2 = 0.17157...$, $\alpha_3 = 0.22138...$, $\alpha_4 = 0.24810...$, $\alpha_5 = 0.26495....$ Kertz [6] proved that α_n is strictly increasing and that

$$\alpha_\infty = \lim_{n\to\infty} \alpha_n = 0.3414889923...$$

is the unique solution of

$$\int_0^1 \frac{1}{u - u\ln(u) + x} du = 1.$$

Hence a prophet may never win more, on average, than 1.34... times the winnings of a mortal.

Suppose instead that you view a sequence X_1, X_2, ..., X_n of independent identically distributed random variables taking values only in the interval $[0,1]$. Everything else is the same. With this additional information, the optimal stopping value V_n now satisfies

$$M_n \leq V_n + \beta_n$$

for best constants β_n with $0 < \beta_n < 1/4$. Again, let us be more precise [5, 7, 8]. Define β_n to be the unique solution of

$$(n-1)\left(g_{n,n}(x) - g_{n-1,n}(x)\right) = 1, \qquad 0 < x < 1.$$

Sample $g_{n-1,n}(x)$ expressions were given earlier; sample $g_{n,n}(x)$ expressions are [5, 7]

$$g_{2,2}(x) = \frac{2}{1}\left(\frac{2}{1}\left(\frac{x}{1}\right)^{1/2} + \frac{x}{1}\right)^{1/2} + \frac{x}{1},$$

$$g_{3,3}(x) = \frac{3}{2}\left(\frac{3}{2}\left(\frac{3}{2}\left(\frac{x}{2}\right)^{2/3} + \frac{x}{2}\right)^{2/3} + \frac{x}{2}\right)^{2/3} + \frac{x}{2},$$

$$g_{4,4}(x) = \frac{4}{3}\left(\frac{4}{3}\left(\frac{4}{3}\left(\frac{4}{3}\left(\frac{x}{3}\right)^{3/4} + \frac{x}{3}\right)^{3/4} + \frac{x}{3}\right)^{3/4} + \frac{x}{3}\right)^{3/4} + \frac{x}{3}$$

and give rise to $\beta_2 = 1/16$, $\beta_3 = 0.07761...$, $\beta_4 = 0.08538....$ It seems likely that β_n is strictly increasing, but a proof that

$$\beta_\infty = \lim_{n\to\infty} \beta_n \approx 0.1113$$

exists is open. A high-precision estimate of β_∞ is also desired.

The nested radical expressions for $g_{k,n}(x)$ deserve more study. A helpful survey on general prophet inequalities [9] is recommended.

References

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