

# Hammersley's Path Process

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The following is a generalization of a process introduced by Hammersley [1, 2]. Fix three parameters  $\lambda > 0$ ,  $\alpha_+ \geq 0$  and  $\alpha_- \geq 0$ . Let  $P(\mu)$  denote a Poisson random variable with mean  $\mu$ . Baik & Rains [3] constructed a set of points  $S$  in the unit square  $[0, 1] \times [0, 1]$  according to three rules:

- $P(\lambda^2)$  points are selected uniformly inside  $(0, 1) \times (0, 1)$
- $P(\alpha_+ \lambda)$  points are selected uniformly on the open bottom edge  $(0, 1) \times \{0\}$
- $P(\alpha_- \lambda)$  points are selected uniformly on the open left edge  $\{0\} \times (0, 1)$ .

These rules are independently executed. No points are selected from the closed top and right edges, nor is the origin  $(0, 0)$  allowed.

Consider any sequence of distinct points of the form

$$(0, 0), (s_1, t_1), (s_2, t_2), \dots, (s_n, t_n), (1, 1)$$

where each  $(s_k, t_k) \in S$ ,  $1 \leq k \leq n$ , and  $n$  is arbitrary. For convenience, define  $(s_0, t_0) = (0, 0)$  and  $(s_{n+1}, t_{n+1}) = (1, 1)$ . Define such a point sequence to be an **up/right path** if, for any  $k \geq 1$ , we have  $s_{k-1} \leq s_k$  and  $t_{k-1} \leq t_k$ . Hence an up/right path joins points of  $S$  in a continuous, piecewise linear manner with line segments of slope  $m_k$ ,  $0 \leq m_k \leq \infty$ , attaching  $(s_{k-1}, t_{k-1})$  and  $(s_k, t_k)$  for all  $k$ .

Of all up/right paths determined by  $S$ , there is (at least) one with a maximum number  $n$  of points. Call this number  $N_\lambda$ . (This is usually referred to as a *length* in the literature. Of course, it also depends implicitly on  $\alpha_+$  and  $\alpha_-$ .) What can be said about the probability distribution of  $N_\lambda$  as  $\lambda \rightarrow \infty$ ?

A special case of the above is the longest increasing subsequence problem [4], achieved when  $\alpha_+ = \alpha_- = 0$ . Its solution will be folded into the formulas we give shortly for the general problem. This turns out to be related to the polynuclear growth (PNG) model in physics due to Prähofer & Spohn [5, 6, 7], but we cannot discuss such topics now.

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When  $0 \leq \alpha_+ \leq 1$  and  $0 \leq \alpha_- \leq 1$  are fixed, the following formulas hold [3]:

$$\lim_{\lambda \rightarrow \infty} \mathbb{P} \left( \frac{N_\lambda - 2\lambda}{\lambda^{1/2}} \leq x \right) = \begin{cases} F_{\text{GUE}}(x) & \text{if } \alpha_+ < 1 \text{ and } \alpha_- < 1, \\ F_{\text{GOE}}(x)^2 & \text{if } \alpha_+ = 1, \alpha_- < 1 \text{ or } \alpha_+ < 1, \alpha_- = 1, \\ F_0(x) & \text{if } \alpha_+ = 1 \text{ and } \alpha_- = 1, \end{cases}$$

where the distribution functions  $F_{\text{GUE}}(x)$ ,  $F_{\text{GOE}}(x)$  and  $F_0(x)$  will be defined shortly. Also, when  $\alpha_+ > 1$  or  $\alpha_- > 1$ , we have

$$\lim_{\lambda \rightarrow \infty} \mathbb{P} \left( \frac{N_\lambda - (\alpha + \alpha^{-1})\lambda}{\sqrt{\alpha - \alpha^{-1}}\lambda^{1/2}} \leq x \right) = \begin{cases} \Phi(x) & \text{if } \alpha_+ \neq \alpha_-, \\ \Phi(x)^2 & \text{if } \alpha_+ = \alpha_-, \end{cases}$$

where  $\alpha = \max\{\alpha_+, \alpha_-\}$  and  $\Phi(x)$  is the standard normal distribution function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{r^2}{2}\right) dr = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}.$$

We provide moments corresponding to these distributions (and more) in Tables 1 and 2; computations were performed by Prähofer [8]. The functions  $F_{\text{GUE}}(x)$ ,  $F_{\text{GOE}}(x)$  and  $F_{\text{GSE}}(x)$  were first discovered by Tracy & Widom [9, 10, 11], whereas  $F_0(x)$  arose more recently [3]. See Figure 1 for the associated density plots.

Table 1. *Moments of GUE, GOE, GUE<sup>2</sup> and GOE<sup>2</sup>*

	$F_{\text{GUE}}$
mean	-1.7710868074...
variance	0.8131947928... = (0.9017731382...) <sup>2</sup>
skewness	0.2240842036...
kurtosis	0.0934480876...
	$F_{\text{GOE}}$
mean	-1.2065335745... = 2 <sup>2/3</sup> (-0.7600685240...)
variance	1.6077810345... = (1.2679830576...) <sup>2</sup> = 2 <sup>4/3</sup> (0.6380483264...)
skewness	0.2934645240...
kurtosis	0.1652429384...
	$F_{\text{GUE}}^2$
mean	-1.2633181526...
variance	0.6066887541... = (0.7789022750...) <sup>2</sup>
skewness	0.3290093382...
kurtosis	0.2254319482...
	$F_{\text{GOE}}^2$
mean	-0.4936399332...
variance	1.2320144032... = (1.1099614422...) <sup>2</sup>
skewness	0.3917246784...
kurtosis	0.3086329720...



Let  $u(x)$  be the solution of the Painlevé II differential equation:

$$u''(x) = 2u(x)^3 + xu(x), \quad u(x) \sim -\frac{1}{2\sqrt{\pi}}x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right) \text{ as } x \rightarrow \infty,$$

and define

$$U(x) = -\int_x^\infty u(r) dr, \quad V(x) = -\int_x^\infty v(r) dr$$

where

$$v(x) = -\int_x^\infty u(r)^2 dr.$$

The largest eigenvalue of a random complex Hermitian matrix, when generated according to the Gaussian Unitary Ensemble (GUE) probability law and properly normalized, has distribution function

$$F_{\text{GUE}}(x) = \exp(-V(x)) \quad (\text{often denoted as the case } \beta = 2).$$

More details appear in [0.1]. Replacing Hermitian matrices by real symmetric matrices, we obtain the Gaussian Orthogonal Ensemble (GOE) and corresponding distribution function

$$F_{\text{GOE}}(x) = \exp\left(-\frac{U(x) + V(x)}{2}\right) \quad (\text{often denoted as the case } \beta = 1).$$

Likewise, for the Gaussian Symplectic Ensemble (GSE), we have

$$F_{\text{GSE}}(x) = \cosh\left(\frac{U(x)}{2}\right) \exp\left(-\frac{V(x)}{2}\right) \quad (\text{the case } \beta = 4).$$

Define also

$$F_0(x) = \left[1 - \left(x + 2u'(x) + 2u(x)^2\right) v(x)\right] \exp(-2U(x) - V(x)),$$

which does not yet seem to possess a random matrix interpretation. These formulas serve as the computational basis for Tables 1 and 2, where skewness and kurtosis of a random variable  $Y$  are given as

$$\text{Skew}(Y) = \frac{\mathbb{E}[(Y - \mathbb{E}(Y))^3]}{\text{Var}(Y)^{3/2}}, \quad \text{Kurt}(Y) = \frac{\mathbb{E}[(Y - \mathbb{E}(Y))^4]}{\text{Var}(Y)^2} - 3.$$

For example, if  $\alpha_+ < 1$  and  $\alpha_- < 1$ , it follows that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1/3}(\mathbb{E}(N_\lambda) - 2\lambda) = -1.7710868074\dots, \quad \lim_{\lambda \rightarrow \infty} \lambda^{-2/3} \text{Var}(N_\lambda) = 0.8131947928\dots$$

which generalize results given earlier by Tracy & Widom and Baik, Deift & Johansson [4]. If instead  $\alpha_+ = 1$  and  $\alpha_- = 1$ , we have

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1/3} \text{Var}(N_\lambda) = 1.1503944782\dots$$

which is called the Baik-Rains constant in [7].

**0.1. GUE/GOE/GSE.** A random complex Hermitian  $N \times N$  matrix  $X$  belongs to GUE if its (real) diagonal elements  $x_{jj}$  and (complex) upper triangular elements  $x_{jk} = \xi_{jk} + i\eta_{jk}$  are independently chosen from zero-mean Gaussian distributions with  $\text{Var}(x_{jj}) = 2$  for  $1 \leq j \leq N$  and  $\text{Var}(\xi_{jk}) = \text{Var}(\eta_{jk}) = 1$  for  $1 \leq j < k \leq N$ . Let  $\lambda$  denote the largest (real) eigenvalue of  $X$  and define the normalization [11]

$$\tilde{\lambda} = \frac{N^{1/6}(\lambda - 2\sigma\sqrt{N})}{\sigma}$$

where  $\sigma = \sqrt{\text{Var}(x_{jk})} = \sqrt{2}$ . Then the distribution of  $\tilde{\lambda}$  has the moments indicated for GUE in Table 1. A related discussion, involving spacings between adjacent eigenvalues of  $X$  and featuring connections to the Riemann hypothesis, appears in [12].

A random real symmetric  $N \times N$  matrix  $X$  belongs to GOE if its diagonal elements  $x_{jj}$  and upper triangular elements  $x_{jk}$  are independently chosen from zero-mean Gaussian distributions with  $\text{Var}(x_{jj}) = 2$  and  $\text{Var}(x_{jk}) = 1$ . Let  $\tilde{\lambda}$  denote the largest (real) eigenvalue of  $X$ , normalized as before with  $\sigma = 1$  in this case. Then the distribution of  $\tilde{\lambda}$  has the moments indicated for GOE in Table 1.

A complex Hermitian  $2N \times 2N$  matrix is said to be **real quaternionic** [13] if, when viewed as an  $N \times N$  matrix  $X$  consisting of  $2 \times 2$  blocks, the diagonal blocks  $X_{jj}$  look like

$$X_{jj} = \begin{pmatrix} x_{jj} & 0 \\ 0 & x_{jj} \end{pmatrix}, \quad x_{jj} \in \mathbb{R}$$

and the upper triangular blocks  $X_{jk}$  look like

$$X_{jk} = \begin{pmatrix} \xi_{jk} + i\eta_{jk} & \xi'_{jk} + i\eta'_{jk} \\ -\xi'_{jk} + i\eta'_{jk} & \xi_{jk} - i\eta_{jk} \end{pmatrix}.$$

A random real quaternionic matrix  $X$  belongs to GSE if the nonzero distinct elements of its diagonal and upper triangular blocks are independently chosen from zero-mean Gaussian distributions with  $\text{Var}(x_{jj}) = 2$  and  $\text{Var}(\xi_{jk}) = \text{Var}(\eta_{jk}) = \text{Var}(\xi'_{jk}) = \text{Var}(\eta'_{jk}) = 1$ . Let  $\tilde{\lambda}$  denote the largest (real) eigenvalue of  $X$ , normalized as before with  $\sigma = 2$  in this case. Then the distribution of  $\tilde{\lambda}$  has the moments indicated for GSE in Table 2.

Here is an occurrence of  $F_{\text{GUE}}(x)^2$ : Define a **signed permutation**  $\pi$  to be a bijection from  $\{-n, -n+1, \dots, -2, -1, 1, 2, \dots, n-1, n\}$  onto itself which satisfies  $\pi(-k) = -\pi(k)$  for all  $k$ . Tracy & Widom [14, 15] proved that the length  $L_{2n}$  of the longest increasing subsequence of a random signed permutation  $\pi$  satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{L_{2n} - 2\sqrt{2n}}{2^{2/3}(2n)^{1/6}} \leq x \right) = F_{\text{GUE}}(x)^2.$$

A nice combinatorial application involving  $F_{\text{GSE}}(x)$  or  $F_{\text{GSE}}(x)^2$ , especially one as simple as this, would be good to find.

Other applications appear in [11, 16, 17]. A  $d$ -dimensional analog of Hammersley's original process (with  $\alpha_+ = \alpha_- = 0$ ) appears in [18]: Let  $S$  denote a set of  $P(\lambda^d)$  points selected uniformly inside the  $d$ -dimensional unit cube and  $N_\lambda$  denote the number of points in a maximal chain (totally ordered subset) of  $S$ . Define  $c_d$  to be  $\limsup_{\lambda \rightarrow \infty} \mathbb{E}(N_\lambda)/\lambda$ . Then it is known that  $c_2 = 2$  and  $c_\infty = e$ , but  $2.363 \leq c_3 \leq 2.366$ ,  $2.514 \leq c_4 \leq 2.521$ ,  $2.583 \leq c_5 \leq 2.589$  and  $2.607 \leq c_6 \leq 2.617$ . We draw attention finally to the obvious identity [2]:

$$F_{\text{GSE}}(x) = \frac{1}{2} \left( F_{\text{GOE}}(x) + \frac{F_{\text{GUE}}(x)}{F_{\text{GOE}}(x)} \right)$$

and wonder whether a similar identity relating  $F_0$  to other distributions can ever be found.

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