

Radii in Geometric Function Theory

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First, we talk about geometry. A region $R \subseteq \mathbb{C}$ is **convex** if, for any two points $p, q \in R$, the line segment $pq \subseteq R$. A region $R \subseteq \mathbb{C}$ is **starlike** with respect to the origin if $0 \in R$ and if, for any point $p \in R$, the line segment $0p \subseteq R$.

Next, we talk about functions. A complex analytic function f defined on an open region is **univalent** (or **schlicht**) if f is one-to-one; that is, $f(z) = f(w)$ if and only if $z = w$. Let

$$D = \{z : |z| < 1\} \quad (\text{the open disk of radius 1}),$$

$$E = \{z : 0 < |z| < 1\} \quad (\text{the open punctured disk}),$$

$$S = \left\{ \text{univalent } f \text{ on } D \text{ with } f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\},$$

$$\Sigma = \left\{ \text{univalent } f \text{ on } E \text{ with } f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \right\}.$$

Geometry and functions now come together. The various subclasses of S include

$$\begin{aligned} CV &= \{f \in S : f(D) \text{ is convex}\} \\ &= \left\{ f \in S : \operatorname{Re} \left(1 + z \frac{f''(z)}{f'(z)} \right) > 0 \text{ for all } z \in D \right\}, \end{aligned}$$

the class of convex functions on D , and

$$\begin{aligned} ST &= \{f \in S : f(D) \text{ is starlike with respect to } 0\} \\ &= \left\{ f \in S : \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > 0 \text{ for all } z \in D \right\}, \end{aligned}$$

the class of starlike functions on D . We will mostly discuss S (the analytic case), but will mention Σ (the meromorphic case) occasionally in the following [1, 2, 3, 4, 5].

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0.1. Radius of Convexity. Define $D_r = \{z : |z| < r\}$, the open disk of radius r , for each $r > 0$. For each $f \in S$, let $r(f)$ be the supremum of all numbers r such that $f(D_r)$ is convex. The **radius of convexity** for S is [1]

$$\rho_{cv}(S) = \inf_{f \in S} r(f) = 2 - \sqrt{3} = 0.2679491924\dots$$

and is achieved by the Koebe function $f(z) = z(1 - z)^{-2}$. This fact was first proved by Nevanlinna [6]. Generalization of ρ_{cv} to any subclass of S gives rise to some interesting optimization problems. Trivially we have

$$\rho_{cv}(CV) = 1, \quad \rho_{cv}(ST) = 2 - \sqrt{3}$$

(the latter follows since the Koebe function is starlike). Define, however, the special class of starlike functions of order α :

$$S_\alpha^* = \left\{ f \in S : \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > \alpha \text{ for all } z \in D \right\}.$$

Zmorovic [7], extending work in [8, 9, 10], proved that

$$\rho_{cv}(S_\alpha^*) = \begin{cases} \frac{1}{2 - 3\alpha + \sqrt{(1 - \alpha)(3 - 5\alpha)}} & \text{if } 0 \leq \alpha < \alpha_0, \\ \left(\frac{5\alpha - 1}{4\alpha^2 - \alpha + 1 + 4\alpha\sqrt{\alpha^2 - 3\alpha + 2}} \right)^{\frac{1}{2}} & \text{if } \alpha_0 \leq \alpha < 1, \end{cases}$$

where $\alpha_0 = 0.3349596751\dots$ is the smallest positive zero of $20\alpha^4 - 52\alpha^3 + 15\alpha^2 + 12\alpha - 4$. Note that $\rho_{cv}(S_0^*) = 2 - \sqrt{3}$, as expected.

We turn attention to the class Σ . Define $E_r = \{z : 0 < |z| < r\}$ and, for $f \in \Sigma$, let $r(f)$ be the supremum of all numbers r such that the complement of $f(E_r)$ in \mathbb{C} is convex. Goluzin [5, 11] proved that

$$\rho_{cv}(\Sigma) = \inf_{f \in \Sigma} r(f) = x = 0.5600798519\dots$$

where x is the unique positive solution of the equation

$$\frac{E(x)}{K(x)} + \frac{x^2}{8} - \frac{7}{8} = 0$$

and $K(x)$, $E(x)$ are complete elliptic integrals of the first and second kind [12]. Letting

$$\Sigma_\beta^* = \left\{ f \in \Sigma : \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) < -\beta \text{ for all } z \in E \right\},$$

we also have [7, 9, 11, 13, 14]

$$\rho_{cv}(\Sigma_\beta^*) = \begin{cases} \left(\frac{4\beta - 5 + 4\sqrt{\beta^2 - \beta + 1}}{8\beta - 3} \right)^{\frac{1}{2}} & \text{if } 0 \leq \beta < \beta_0, \\ \frac{1}{\beta + \sqrt{(1 - \beta)(3\beta - 1)}} & \text{if } \beta_0 \leq \beta < 1, \end{cases}$$

where $\beta_0 = 0.8673407553\dots$ is the largest positive zero of $12\beta^4 - 28\beta^3 + 33\beta^2 - 20\beta + 4$. Note here that $\rho_{cv}(\Sigma_0^*) = 1/\sqrt{3} = 0.577\dots > 0.560\dots = x$. In this case, the extremal function is not starlike, which accounts for the strict inequality.

0.2. Radius of Starlikeness. For each $f \in S$, let $r(f)$ be the supremum of all numbers r such that $f(D_r)$ is starlike with respect to the origin. The **radius of starlikeness** for S is [1]

$$\rho_{st}(S) = \inf_{f \in S} r(f) = \frac{1 - e^{-\pi/2}}{1 + e^{-\pi/2}} = \tanh\left(\frac{\pi}{4}\right) = 0.6557942026\dots$$

and this fact was first discovered by Grunsky [15].

Goluzin [5, 16] found several interesting generalizations. Define a region $R \subseteq \mathbb{C}$ to be **n -starlike** with respect to the origin if $0 \in R$ and if every point of R can be connected with 0 by a piecewise linear curve that lies entirely in R and that consists of no more than n line segments. Let δ_n be the supremum of all r such that an arbitrary $f \in S$ maps D_r onto an n -starlike region with respect to 0. Then

$$\tanh\left(\frac{\pi}{4}\right) = \delta_1 \leq \delta_2 \leq \delta_3 \leq \dots, \quad \delta_n \geq \tanh\left(\frac{n\pi}{4}\right),$$

but values for δ_n , $n \geq 2$, are unknown. See also [17, 18].

Likewise, let ϵ_n be the supremum of all r such that an arbitrary $f \in \Sigma$ maps E_r onto a region, the complement of which is n -starlike with respect to 0. Then

$$0.85 < \epsilon_1, \quad 1 - 1.11 \exp\left(\frac{-n\pi}{2}\right) < \epsilon_n \quad \text{for all } n > 1.$$

An exact expression for ϵ_1 would be good to see someday.

0.3. Radius of Close-to-Convexity. A region $R \subseteq \mathbb{C}$ is **close-to-convex** (or **linearly accessible**) if its complement is a union of closed half-lines such that the corresponding open half-lines are pairwise disjoint. Any starlike region is close-to-convex. A half-annulus is also close-to-convex, but this property fails for any larger subsection of an annulus.

An analytic function $f : D \rightarrow \mathbb{C}$ is close-to-convex if $f(D)$ is close-to-convex. Equivalently, f is close-to-convex if there is a convex function $g : D \rightarrow \mathbb{C}$ such that $\operatorname{Re}(f'(z)/g'(z)) > 0$ for all $z \in D$ [1, 19, 20, 21, 22, 23, 24, 25]. It can be shown that every close-to-convex function is univalent.

Define

$$\begin{aligned}
 CC &= \{f \in S : f(D) \text{ is close-to-convex}\} \\
 &= \left\{ f \in S : \int_{\theta_1}^{\theta_2} \operatorname{Re} \left(1 + z \frac{f''(z)}{f'(z)} \right) d\theta > -\pi, \text{ where } z = re^{i\theta}, \right. \\
 &\quad \left. \text{for each } 0 < r < 1 \text{ and each pair } 0 < \theta_1 < \theta_2 < 2\pi \right\}.
 \end{aligned}$$

Let $\rho_{cc}(S)$ be the supremum of all r such that an arbitrary $f \in S$ maps D_r onto a close-to-convex region. Krzyz [26] determined that

$$\rho_{cc}(S) = y = 0.8098139153\dots$$

where y is the unique real solution of the equation

$$2 \arctan \left(\frac{\kappa(y)}{\lambda(y)} \right) + \ln \left(1 + \lambda(y)^2 \right) - 2 \ln \left(\frac{2y}{1 - y^2} \right) = 0$$

in the interval $0 < y < 1$, $\kappa(y) = (1 + y^2)/(1 - y^2)$, and $\lambda = \lambda(y)$ is the unique real solution of the equation

$$\lambda^3 - \kappa(y)\lambda^2 + \kappa(y)^2\lambda - \kappa(y) = 0.$$

Sizuk [27] extended this result to the class of close-to-convex functions of order γ .

0.4. Radius of Convexity in One Direction. A region $R \subseteq \mathbb{C}$ is **convex in the direction of the imaginary axis** if, for every vertical line L , the set $L \cap R$ is either empty or connected. Any region that is convex in one direction can be rotated so that it is convex in the imaginary direction [3, 28, 29].

Define

$$CD = \{f \in S : f(D) \text{ is convex in the imaginary direction}\}$$

and let $\rho_{cd}(S)$ be the supremum of all numbers r such that an arbitrary $f \in S$ maps D_r onto a region that is convex in the imaginary direction. Umezawa [30] and Goodman & Saff [31] proved that

$$0.394\dots = 4 - \sqrt{13} \leq \rho_{cd}(S) \leq \sqrt{2} - 1 = 0.414\dots$$

The exact value of this constant is unknown.

A subclass of CD was considered by Hengartner & Schober [32]:

$$\{f \in S : \operatorname{Re} \left((1 - z^2)f'(z) \right) \geq 0 \text{ for all } z \in D\}$$

but we omit details. See also [33, 34].

0.5. Radius of Majorization. Let $f : D \rightarrow \mathbb{C}$ be analytic with $f(0) = 0$ and $f'(0) \geq 0$. Let $F \in S$. The function f is **subordinate** to F , written $f \preceq F$, if $f(D_r) \subseteq F(D_r)$ for all $0 < r < 1$ [1, 35].

Shah [36, 37], verifying conjectures of Goluzin [5, 38], proved that if $f \preceq F$, then

$$|f(z)| \leq |F(z)| \quad \text{for all } |z| \leq \frac{1}{2}(3 - \sqrt{5}) = 0.3819660112\dots,$$

$$|f'(z)| \leq |F'(z)| \quad \text{for all } |z| \leq 3 - 2\sqrt{2} = 0.1715728752\dots$$

Both of these radii are best possible. If we further assume that f is univalent and $f'(0) > 0$, then [5, 39]

$$|f(z)| \leq |F(z)| \quad \text{for all } |z| \leq u = 0.3908507887\dots,$$

where u is the unique real solution of

$$\ln\left(\frac{1+u}{1-u}\right) + 2\arctan(u) = \frac{\pi}{2}.$$

Again, this radius of majorization is best possible. Problems as such (subordination implies majorization) were first examined by Biernacki [40].

Converse problems (majorization implies subordination) were studied by Lewandowski [41]. Under the same conditions as earlier, if $|f(z)| \leq |F(z)|$ for all $z \in D$ and f is not necessarily univalent, then $f \preceq F$ in the disk D_v , where $0.21 < v < 0.29$. The exact value of v is unknown. If f is assumed to be univalent, then the constant $u = 0.390\dots$ arises again [42, 43].

0.6. Radius of Zeroness. Let $\rho_N(\Sigma)$ be the supremum of all numbers r such that an arbitrary $f \in \Sigma$ never vanishes on the punctured disk E_r . Goluzin [16] proved that $0.86 < \rho_N(\Sigma) \leq \sqrt{3}/2 < 0.867$, but a subsequent theorem of his [5, 44] implies that $\rho_N(\Sigma) = \xi = 0.8649789576\dots$, where ξ is the unique positive solution of the equation

$$\frac{E(\xi)}{K(\xi)} + \frac{\xi^2}{4} - \frac{3}{4} = 0.$$

This is quite similar to the equation prescribed earlier for the radius of convexity $\rho_{cv}(\Sigma)$.

Given an analytic function f , we may likewise define $\rho_N(f)$ to be the supremum of all numbers r such that f , when restricted to E_r , is never zero. For example,

$$\rho_N(f) = 2|z_0| \quad \text{for } f(z) = z - \frac{1}{2z_0}z^2 \quad (\text{a quadratic function})$$

and

$$\rho_N(f) = 2\pi \quad \text{for } f(z) = \exp(z) - 1 \quad (\text{the exponential function}).$$

0.7. Radius of Univalence. Given an analytic function f , define the **radius of univalence** of f to be the supremum of all numbers r such that f , when restricted to the disk D_r , is univalent. Let us first consider the case of polynomials. We clearly have

$$\rho_s(f) = |z_0| \quad \text{for } f(z) = z - \frac{1}{2z_0}z^2$$

in the quadratic case. Kakeya's theorem [45, 46, 47] provides that

$$\sin\left(\frac{\pi}{n}\right) \leq \frac{\rho_s(f)}{|z_0|} \leq 1 \quad \text{for } f(z) = z + \sum_{k=2}^n a_k z^k$$

in the general case, where $n \geq 2$ and $z_0 \neq 0$ is the zero of $f'(z)$ of smallest modulus. These bounds are sharp.

Now, let us consider the case of transcendental functions. We have

$$\rho_s(f) = \pi \quad \text{for } f(z) = \exp(z) - 1,$$

as is well-known (although $f'(z)$ never vanishes); [48]

$$\rho_s(f) = 1.5748375891\dots \quad \text{for } f(z) = \operatorname{erf}(z),$$

corresponding to the smallest modulus, of points z not on the x -axis, for which $\operatorname{erf}(z)$ is real (see [49] for definition); [50, 51]

$$\rho_s(f) = 0.9241388730\dots \quad \text{for } f(z) = \exp(z^2) \operatorname{erf}(z),$$

corresponding to the unique positive solution of $\sqrt{\pi}y \operatorname{Im}(f(iy)) = 1$; [52, 53, 54]

$$\rho_s(f) = p_{\nu,1} \quad \text{for } f(z) = z^{1-\nu} J_\nu(z), \quad \nu > -1,$$

corresponding to the smallest positive zero of $f'(z)$ (see [55] for numerical values); [56]

$$\rho_s(f) = 0.5040830082\dots \quad \text{for } f(z) = 1/\Gamma(z),$$

corresponding to the smallest positive zero of $\Gamma'(-z)$; and [57]

$$\rho_s(f) = 0.4616321449\dots \quad \text{for } f(z) = \Gamma(z+1),$$

corresponding to the smallest positive zero of $\Gamma'(z+1)$. See also [58].

We digress briefly to other radii. For $f(z) = \exp(z) - 1$, it is known that [59, 60]

$$\rho_{cv}(f) = 1, \quad \rho_{st}(f) = 2.8329700604\dots$$

and the latter corresponds to $\sqrt{1+\eta^2}$, where η is the smallest positive solution of the equation

$$\eta \sin(\eta) + \cos(\eta) = \frac{1}{e}.$$

See also [61, 62].

0.8. Sums and Products. Here are two procedures for combining univalent functions:

$$S + S = \{h : h(z) = tf(z) + (1-t)g(z) \text{ for some } f, g \in S \text{ and } 0 \leq t \leq 1\},$$

$$S \cdot S = \left\{h : h(z) = f(z)^t g(z)^{1-t} \text{ for some } f, g \in S \text{ and } 0 \leq t \leq 1\right\}.$$

On the one hand, MacGregor [63] demonstrated that

$$\rho_s(S + S) = \sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0.3826834323\dots$$

$$\rho_s(CV + CV) = \frac{\sqrt{2}}{2} = 0.7071067811\dots$$

and Robertson [64] showed that

$$\rho_s(ST + ST) = \chi = 0.4035150049\dots$$

where χ is the unique positive zero of $\chi^6 + 5\chi^4 + 79\chi^2 - 13$. Further results appear in [65, 66, 67]. On the other hand, we have [3]

$$CV \cdot CV \subseteq ST \cdot ST \subseteq ST, \quad CV \cdot CV \not\subseteq CV$$

but virtually nothing is known about the class $S \cdot S$.

0.9. Derivatives and Integrals. Define the following classes of functions:

$$T = \left\{f : f(z) = \frac{1}{2} \frac{d}{dz} (zg(z)) \text{ for some } g \in S\right\},$$

$$U_\alpha = \left\{f : f(z) = \int_0^z \left(\frac{g(w)}{w}\right)^\alpha dw \text{ for some } g \in S\right\},$$

$$V_\beta = \left\{f : f(z) = \int_0^z g'(w)^\beta dw \text{ for some } g \in S\right\},$$

where α, β are complex numbers and hence the logarithmic branch is selected so that $f'(0) = 1$. Barnard [68, 69] and Pearce [70], building on Robinson [71], proved that

$$0.49 < \rho_s(T) \leq \frac{1}{2}, \quad 0.435 < \rho_{st}(T) < 0.445.$$

In particular, these two constants must be distinct.

Biernacki [72] claimed that $\rho_s(U_1) = 1$, but this was disproved by Krzyz & Lewandowski [73]. It was later shown [74] that $0.91 < \rho_s(U_1) \leq \tanh(\pi) < 0.9963$. Let A denote the set of all complex numbers α for which $U_\alpha \subseteq S$. Kim & Merkes [75] proved that $D_{1/4} \subseteq A \subseteq D_{1/2}$; we wonder whether $D_r \subseteq A$ for some $r > 1/4$.

Trivially $\rho_s(V_1) = 1$. Let B denote the set of all complex numbers β for which $V_\beta \subseteq S$. Royster [76] and Pfaltzgraff [77] proved that $D_{1/4} \subseteq B \subseteq D_{1/3} \cup \{1\}$; we again wonder whether $D_r \subseteq B$ for some $r > 1/4$. See also [78, 79].

REFERENCES

- [1] P. L. Duren, *Univalent Functions*, Springer-Verlag, 1983, pp. 40–51, 95–98, 202–209; MR0708494 (85j:30034).
- [2] A. W. Goodman, *Univalent Functions*, v. 1, Mariner, 1983, pp. 119–121, 193–197; MR0704183 (85j:30035a).
- [3] A. W. Goodman, *Univalent Functions*, v. 2, Mariner, 1983, pp. 84–172; MR0704184 (85j:30035b).
- [4] W. K. Hayman, *Multivalent Functions*, 2nd ed., Cambridge Univ. Press, 1994, pp. 226–227; MR1310776 (96f:30003).
- [5] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable* (in Russian), 2nd ed., Izdat. Akad. Nauk SSSR, 1966; Engl. transl. in *Amer. Math. Soc. Transl.*, v. 26, 1969, pp. 136–140, 165–170, 375–379, 599, 643–644; MR0219714 (36 #2793) and MR0247039 (40 #308).
- [6] R. Nevanlinna, Über die Eigenschaften einer analytischen Funktion in der Umgebung einer singulären Stelle oder Linie, *Acta Soc. Sci. Fenn.*, v. 50 (1922) n. 5, 1–46.
- [7] V. A. Zmorovic, On bounds of convexity for starlike functions of order α in the circle $|z| < 1$ and in the circular region $0 < |z| < 1$ (in Russian), *Mat. Sbornik* 68 (1965) 518–526; Engl. transl. in *Amer. Math. Soc. Transl.*, v. 80, 1969, pp. 203–213; MR0197712 (33 #5875).
- [8] T. H. MacGregor, The radius of convexity for starlike functions of order $\frac{1}{2}$, *Proc. Amer. Math. Soc.* 14 (1963) 71–76; MR0150282 (27 #283).
- [9] M. S. Robertson, Some radius of convexity problems, *Michigan Math. J.* 10 (1963) 231–236; MR0151596 (27 #1580).
- [10] A. Schild, On starlike functions of order α , *Amer. J. Math.* 87 (1965) 65–70; MR0174729 (30 #4929).
- [11] M. S. Robertson, Extremal problems for analytic functions with positive real part and applications, *Trans. Amer. Math. Soc.* 106 (1963) 236–253; MR0142756 (26 #325).
- [12] S. R. Finch, Archimedes' constant: Elliptic functions, *Mathematical Constants*, Cambridge Univ. Press, 2003, p. 26.

- [13] K. S. Padmanabhan, On the radius of convexity of a certain class of meromorphically starlike functions in the unit circle, *Math. Z.* 91 (1966) 308–313; MR0190319 (32 #7732).
- [14] V. Singh and R. M. Goel, On radii of convexity and starlikeness of some classes of functions, *J. Math. Soc. Japan* 23 (1971) 323–339; MR0281903 (43 #7617).
- [15] H. Grunsky, Zwei Bemerkungen zur konformen Abbildung, *Jahresbericht Deutsch. Math.-Verein.* 43 (1934) 140–143.
- [16] G. M. Goluzin, Zur Theorie der schlichten konformen Abbildungen (in Russian), *Mat. Sbornik* 42 (1935) 169–190.
- [17] S. Gelfer, Sur les bornes de l'étoilement et de la convexité des fonctions p -valentes (in Russian), *Mat. Sbornik* 16 (1945) 81–86; MR0012662 (7,55a).
- [18] I. A. Aleksandrov, Domains of definition of some functionals on the class of functions univalent and regular in a circle (in Russian), *Issledovaniia Po Sovremennym Problemam Teorii Funkcii Kompleksnovo Peremennovo*, Gosudarstv. Izdat. Fiz.-Mat. Lit., 1960, pp. 39–45; MR0114928 (22 #5744).
- [19] M. Biernacki, Sur la représentation conforme des domaines linéairement accessibles, *Prace Matematyczno-Fizyczne* 44 (1936) 293–314.
- [20] W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.* 1 (1952) 169–185 (1953); MR0054711 (14,966e).
- [21] Z. Lewandowski, Sur l'identité de certaines classes de fonctions univalentes. I, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 12 (1958) 131–146; MR0130354 (24 #A217).
- [22] Z. Lewandowski, Sur l'identité de certaines classes de fonctions univalentes. II, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 14 (1960) 19–46; MR0156958 (28 #200).
- [23] A. Bielecki and Z. Lewandowski, Sur un théorème concernant les fonctions univalentes linéairement accessibles de M. Biernacki, *Ann. Polon. Math.* 12 (1962) 61–63; MR0147636 (26 #5151).
- [24] V. A. Pohilevic, The equivalence of two classes of univalent functions (in Russian), *Teor. Funkcii Funkcional Anal. i Prilozhen* 8 (1969) 57–62; MR0262482 (41 #7088).

- [25] W. Koepf, On close-to-convex functions and linearly accessible domains, *Complex Variables Theory Appl.* 11 (1989) 269–279; MR1007662 (91f:30010).
- [26] J. Krzyz, The radius of close-to-convexity within the family of univalent functions, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 10 (1962) 201–204; MR0148887 (26 #6384).
- [27] P. I. Sizuk, The radius of almost convexity of order α in a class of schlicht functions (in Russian), *Mat. Zametki* 20 (1976) 105–112; English transl. in *Math. Notes* 20 (1976) 616–620; MR0425103 (54 #13061).
- [28] L. Fejér, Neue Eigenschaften der Mittelwerte bei den Fourierreihen, *J. London Math. Soc.* 8 (19133) 53–62.
- [29] M. S. Robertson, On the theory of univalent functions, *Annals of Math.* 37 (1936) 374–408.
- [30] T. Umezawa, Analytic functions convex in one direction, *J. Math. Soc. Japan* 4 (1952) 194–202; MR0051313 (14,461b).
- [31] A. W. Goodman and E. B. Saff, On univalent functions convex in one direction, *Proc. Amer. Math. Soc.* 73 (1979) 183–187; MR0516461 (80e:30006).
- [32] W. Hengartner and G. Schober, Analytic functions close to mappings convex in one direction, *Proc. Amer. Math. Soc.* 28 (1971) 519–524; MR0277704 (43 #3437).
- [33] R. M. Goel, On a class of analytic functions, *Indian J. Math.* 17 (1975) 9–20; MR0603705 (82i:30011).
- [34] M. Dorff, I. Naraniecka and J. Szynal, Doubly close-to-convex functions, *J. Math. Anal. Appl.*, to appear (2004).
- [35] D. M. Campbell, Majorization-subordination theorems for locally univalent functions, *Bull. Amer. Math. Soc.* 78 (1972) 535–538; MR0299769 (45 #8817).
- [36] T.-S. Shah, Goluzin's number $(3 - \sqrt{5})/2$ is the radius of superiority in subordination, *Sci. Record* 1 (1957) 219–222; MR0100094 (20 #6530).
- [37] T.-S. Shah, On the radius of superiority in subordination, *Sci. Record* 1 (1957) 329–333; MR0100095 (20 #6531).
- [38] G. M. Goluzin, On majorants of subordinate analytic functions. I (in Russian), *Mat. Sbornik* 29 (1951) 209–224; MR0043209 (13,223b).

- [39] G. M. Goluzin, On majoration of subordinate analytic functions. II (in Russian), *Mat. Sbornik* 29 (1951) 593–602; MR0044644 (13,454d).
- [40] M. Biernacki, Sur les fonctions univalentes, *Mathematica (Cluj)* 12 (1936) 49–64.
- [41] Z. Lewandowski, Sur les majorantes des fonctions holomorphes dans le cercle $|z| < 1$, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 15 (1961) 5–11; MR0142745 (26 #314).
- [42] Z. Lewandowski, Starlike majorants and subordination, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 15 (1961) 79–84; MR0138743 (25 #2186).
- [43] A. Bielecki and Z. Lewandowski, Sur certaines majorantes des fonctions holomorphes dans le cercle unité, *Colloq. Math.* 9 (1962) 299–303; MR0152650 (27 #2625).
- [44] G. M. Goluzin, Zur Theorie der schlichten Funktionen (in Russian), *Mat. Sbornik* 12 (1943) 48–55; MR0009053 (5,93f).
- [45] S. Kakeya, On zeros of a polynomial and its derivatives, *Tôhuku Math. J.* 11 (1917) 5–16.
- [46] L. N. Cakalov, On domains of univalence of certain classes of analytic functions (in Russian), *Dokl. Akad. Nauk SSSR* 132 (1960) 1277–1279; English transl. in *Soviet Math. Dokl.* 1 (1960) 781–783; MR0122965 (23 #A296).
- [47] A. Gluchoff and F. Hartmann, On a “much underestimated” paper of Alexander, *Arch. Hist. Exact Sci.* 55 (2000) 1–41; MR1780441 (2001i:30001).
- [48] E. Kreyszig and J. Todd, The radius of univalence of the error function, *Numer. Math.* 1 (1959) 78–89; MR0101918 (21 #724).
- [49] S. R. Finch, Fransén-Robinson constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 262–264.
- [50] E. Kreyszig and J. Todd, On the radius of univalence of the function $\exp z^2 \int_0^z \exp(-t^2) dt$, *Pacific J. Math.* 9 (1959) 123–127; MR0107032 (21 #5759).
- [51] Y. L. Luke, The radius of univalence of the function $\exp z^2 \int_0^z \exp(-t^2) dt$, *Numer. Math.* 3 (1961) 76–78; MR0118844 (22 #9613).
- [52] E. Kreyszig and J. Todd, The radius of univalence of Bessel functions. I, *Illinois J. Math.* 4 (1960) 143–149; MR0110827 (22 #1695).

- [53] R. K. Brown, Univalence of Bessel functions, *Proc. Amer. Math. Soc.* 11 (1960) 278–283; MR0111846 (22 #2706).
- [54] H. S. Wilf, The radius of univalence of certain entire functions, *Illinois J. Math.* 6 (1962) 242–244; MR0138734 (25 #2177).
- [55] S. R. Finch, Bessel function zeroes, unpublished note (2003).
- [56] E. P. Merkes, M. S. Robertson and W. T. Scott, On products of starlike functions, *Proc. Amer. Math. Soc.* 13 (1962) 960–964; MR0142747 (26 #316).
- [57] E. P. Merkes and M. Salmassi, On univalence of certain infinite products, *Complex Variables Theory Appl.* 33 (1997) 207–215; MR1624939 (99b:30023).
- [58] G. V. Kuzmina, Numerical determination of radii of univalence of analytic functions (in Russian), *Trudy Mat. Inst. Steklov.* 53 (1959) 192–235; MR0118841 (22 #9610).
- [59] P. T. Mocanu, Sur le rayon de stellarité des fonctions univalentes (in Romanian), *Acad. R. P. Romène Fil. Cluj Stud. Cerc. Mat.* 11 (1960) 337–341; MR0146371 (26 #3893).
- [60] P. T. Mocanu, About the radius of starlikeness of the exponential function, *Studia Univ. Babeş-Bolyai Ser. Math.-Phys.* 14 (1969) 35–40; MR0257339 (41 #1990).
- [61] D. Coman, The radius of starlikeness for the error function, *Studia Univ. Babeş-Bolyai Math.* 36 (1991) 13–16; MR1280904.
- [62] D. Coman and I. Tigan, The close-to-convexity radius of some functions, *Studia Univ. Babeş-Bolyai Math.* 36 (1991) 61–65; MR1281977.
- [63] T. H. MacGregor, The univalence of a linear combination of convex mappings, *J. London Math. Soc.* 44 (1969) 210–212; MR0236369 (38 #4665).
- [64] M. S. Robertson, The sum of univalent functions, *Duke Math. J.* 37 (1970) 411–419; MR0264052 (41 #8649).
- [65] G. Labelle and Q. I. Rahman, Remarque sur la moyenne arithmétique de fonctions univalentes convexes, *Canad. J. Math.* 21 (1969) 977–981; MR0245775 (39 #7081).
- [66] D. M. Campbell, The radius of convexity of a linear combination of functions in K , $CV_k(\beta)$, S or U_α , *Canad. J. Math.* 25 (1973), 982–985; MR0328045 (48 #6387).

- [67] D. H. Hamilton and P. D. Tuan, Radius of starlikeness of convex combinations of univalent starlike functions, *Proc. Amer. Math. Soc.* 78 (1980) 56–58; MR0548084 (81f:30007).
- [68] R. W. Barnard, On the radius of starlikeness of $(zf)'$ for f univalent, *Proc. Amer. Math. Soc.* 53 (1975) 385–390; MR0382615 (52 #3497).
- [69] R. W. Barnard, On Robinson's $\frac{1}{2}$ conjecture, *Proc. Amer. Math. Soc.* 72 (1978) 135–139; MR0503547 (80j:30014).
- [70] K. Pearce, A note on a problem of Robinson, *Proc. Amer. Math. Soc.* 89 (1983) 623–627; MR0718985 (85h:30014).
- [71] R. M. Robinson, Univalent majorants, *Trans. Amer. Math. Soc.* 61 (1947) 1–35; MR0019114 (8,370e).
- [72] M. Biernacki, Sur l'intégrale des fonctions univalentes, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 8 (1960) 29–34; MR0117341.
- [73] J. Krzyz and Z. Lewandowski, On the integral of univalent functions, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 11 (1963) 447–448; MR0153830 (27 #3791).
- [74] Z. Lewandowski, On a problem of M. Biernacki, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 17 (1963) 39–41 (1965); MR0196055 (33 #4249).
- [75] Y. J. Kim and E. P. Merkes, On an integral of powers of a spirallike function, *Kyungpook Math. J.* 12 (1972) 249–252; MR0320295 (47 #8834).
- [76] W. C. Royster, On the univalence of a certain integral, *Michigan Math. J.* 12 (1965) 385–387; MR0183866 (32 #1342).
- [77] J. A. Pfaltzgraff, Univalence of the integral of $f'(z)^\lambda$, *Bull. London Math. Soc.* 7 (1975) 254–256; MR0387570 (52 #8410).
- [78] D. V. Prokhorov and J. Szynal, On the radius of univalence for the integral of $f'(z)^\alpha$, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 33 (1979) 157–163; MR0689595 (84c:30034).
- [79] M. Dorff and J. Szynal, Linear invariance and integral operators of univalent functions, submitted (2002).