Fractional Parts of Bernoulli Numbers

Steven Finch

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The Bernoulli numbers B_0, B_1, B_2, ... are defined via [1, 2, 3]

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

and satisfy $B_0 = 1$, $B_1 = -1/2$, $(-1)^{k+1} B_{2k} > 0$ and $B_{2k+1} = 0$ for $k \geq 1$. It can be shown that $|B_{2k}|$ is strictly increasing after its minimum at $B_6 = 1/42$, and

$$|B_{2k}| \sim \frac{2(2k)!}{(2\pi)^{2k}} \sim 4\sqrt{\pi k} \left(\frac{k}{e\pi} \right)^{2k}$$

as $k \to \infty$. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of a real number x; for example,

$$\{B_2\} = \{\tfrac{1}{6}\} = \tfrac{1}{6}, \qquad \{B_4\} = \{-\tfrac{1}{30}\} = \tfrac{29}{30},$$
$$\{B_{14}\} = \{\tfrac{7}{6}\} = \tfrac{1}{6}, \qquad \{B_{16}\} = \{-\tfrac{3617}{510}\} = \tfrac{463}{510}.$$

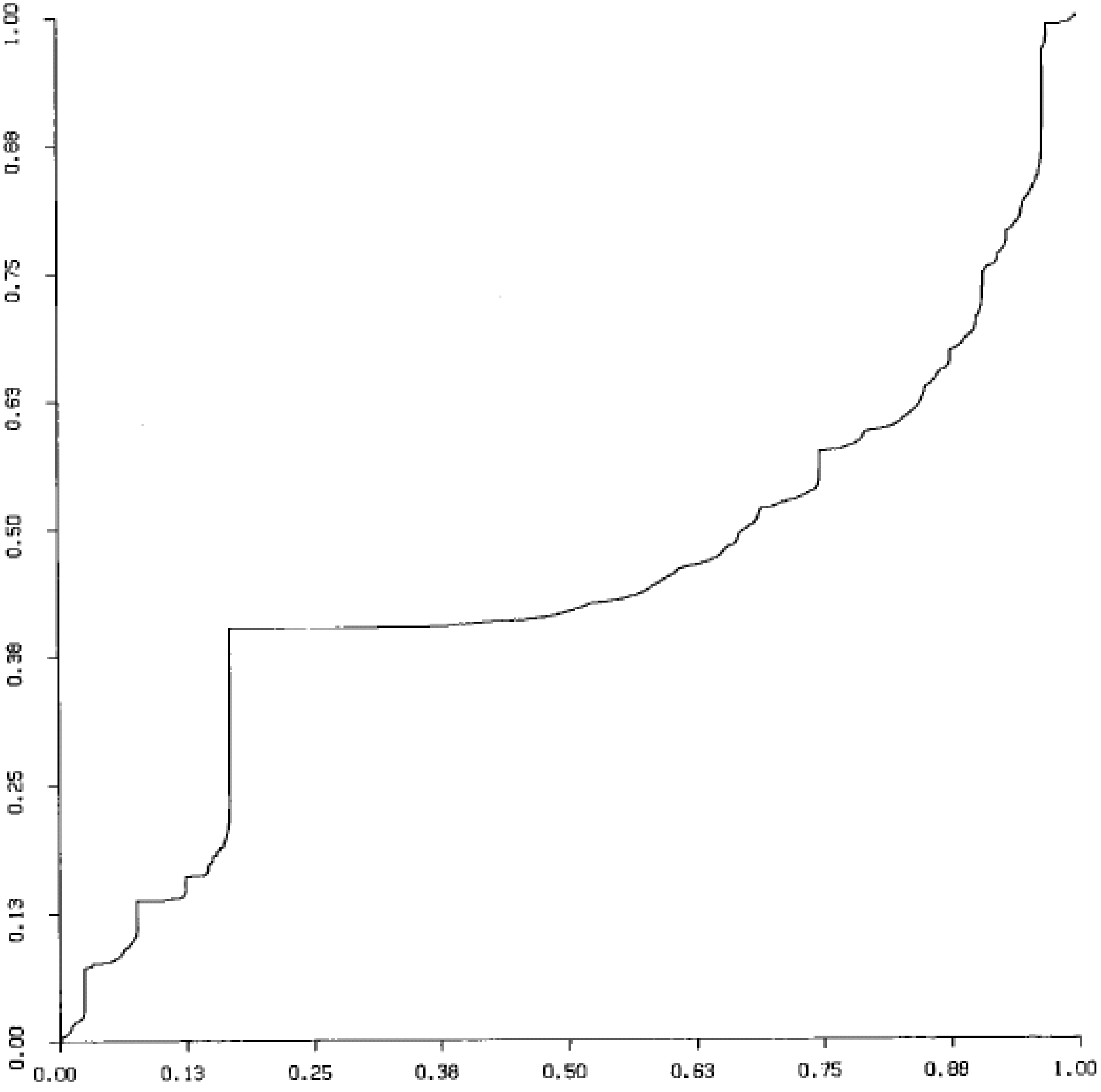
The sequence $\{B_2\}$, $\{B_4\}$, $\{B_6\}$, ... is dense in the unit interval $[0, 1]$, but it is not uniformly distributed [4]. Certain rational numbers appear with positive probability: 1/6 is most likely with probability 0.151..., 29/30 is next with probability 0.064... [5]. In fact, the limiting distribution F is piecewise linear with countably many jump discontinuities: F increases only when jumping (see Figure 1). We wonder, in particular, about the moments of F. By the von Staudt-Clausen theorem, the mean fractional part is [6]

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left\{ -\sum_{(p-1)|2n} \frac{1}{p} \right\} = 0.5486...$$

and the mean fractional part squared is

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left\{ -\sum_{(p-1)|2n} \frac{1}{p} \right\}^2 = 0.4396....$$

Bernoulli Numbers Fractional Parts Distribution



The inner sum is over all primes p such that $p-1$ divides $2n$. No analytic simplification of such formulas is known.

We wonder too about an unrelated quantity

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{p|n}\frac{1}{p}=0.452...$$

which is close to $\sum 1/p^2 = 0.4522474200...$ [7]. Might these two quantities be equal? If the sum $\sum 1/p$ is replaced by the reciprocal of the least prime factor $P^-(n)$ of n, then interestingly [8, 9]

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{P^-(n)}=\sum_{p}\frac{1}{p^2}\prod_{q<p}\left(1-\frac{1}{q}\right)$$

where the inner product is over all primes q less than p. In principle, this latter expression can be evaluated to high precision. A similar replacement for the average of $\{B_{2n}\}$ is not clear. Observe that $p=2$ and $p=3$ both satisfy $(p-1)|2n$ automatically for any $n\geq 1$. The issue is thus determining the smallest such prime exceeding 3 for each n (if one exists) and this may be awkward.

A famous conjecture, due to Siegel [10, 11, 12, 13], is as follows. An odd prime p is **regular** if it does not divide the numerator of any of the Bernoulli numbers B_2, B_4, B_6, ..., B_{p-3}; otherwise p is **irregular**. It seems to be true that

$$\lim_{N\to\infty}\frac{\displaystyle\sum_{\substack{p\leq N,\\ p\text{ irregular}}}1}{\displaystyle\sum_{\substack{p\leq N,\\ p\text{ regular}}}1}=e^{1/2}-1=0.6487212707...$$

but a proof is not known. Equivalently, we have

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\sum_{\substack{p\leq N,\\ p\text{ irregular}}}1=1-e^{-1/2}=0.3934693402...,$$

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\sum_{\substack{p\leq N,\\ p\text{ regular}}}1=e^{-1/2}=0.6065306597....$$

In 1851, Kummer proved that Fermat's Last Theorem holds when the exponent is a regular prime. Although FLT was proved by Wiles in 1995, we still do not know whether there exist infinitely many regular primes.

See also [14, 15] for the asymptotics for $\prod_{k\leq K}|B_{2k}|$.

0.1. Addendum.

Tanguy Rivoal was so kind to answer my question regarding 0.452... with an affirmative proof. Letting

$$S_N = \sum_{n\leq N}\sum_{p|n}\frac{1}{p},$$

it is clear that

$$S_N = \sum_{p\leq N}\frac{1}{p}\sum_{\substack{n\leq N,\\ p|n}} 1 = \sum_{p\leq N}\frac{1}{p}\sum_{m\leq N/p} 1 = \sum_{p\leq N}\frac{\lfloor N/p\rfloor}{p}.$$

Since $N/p - 1 < \lfloor N/p\rfloor \leq N/p$, we obtain

$$\sum_{p\leq N}\frac{1}{p^2} - \frac{1}{N}\sum_{p\leq N}\frac{1}{p} < \frac{1}{N}S_N \leq \sum_{p\leq N}\frac{1}{p^2}$$

and the result follows because $\sum_{p\leq N} 1/p = O(\ln\ln N)$.

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