Dedekind Eta Products

Steven Finch

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For $\mathrm{Im}(z) > 0$, define the **Dedekind eta function**

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \Delta(z)^{1/24}$$

where $q = e^{2\pi i z}$ and $\Delta(z)$ is the discriminant function studied earlier [1]. Euler's pentagonal-number theorem states that

$$\eta(24z) = q \prod_{n=1}^{\infty} \left(1 - q^{24n}\right) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(6k+1)^2};$$

we also have

$$\eta(8z)^3 = q \prod_{n=1}^{\infty} \left(1 - q^{8n}\right)^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{(2k+1)^2}$$

via Jacobi's triple-product identity. The absence of a corresponding formula for

$$\eta(12z)^2 = q \prod_{n=1}^{\infty} \left(1 - q^{12n}\right)^2$$

or for $\eta(8z)\eta(16z)$, $\eta(6z)\eta(18z)$, $\eta(4z)\eta(20z)$, $\eta(3z)\eta(21z)$, $\eta(2z)\eta(22z)$, $\eta(z)\eta(23z)$ is remarkable! At a minimum, we should be able to say something about the density of nonzero coefficients in the q-series expansion (on the right-hand side).

Given any eta product

$$\eta(b_1 z)\eta(b_2 z) \cdots \eta(b_m z) = \sum_{k=0}^{\infty} a_k q^k, \qquad 1 \le b_1 \le b_2 \le \ldots \le b_m,$$

define the counting function

$$M_{b_1,b_2,\ldots,b_m}(x) = \#\left\{k \le x : a_k \ne 0\right\}.$$

The eta product is said to be **lacunary** if

$$\lim_{x\to\infty}\frac{M_{b_1,b_2,\ldots,b_m}(x)}{x}=0.$$

For example, it is clear that

$$M_{24}(x)\sim\frac{1}{3}\sqrt{x},\qquad M_{8,8,8}(x)\sim\frac{1}{2}\sqrt{x}$$

as $x\to\infty$. Serre [2, 3, 4] proved that

$$M_{12,12}(x)\sim\frac{c\,x}{(\ln x)^{3/4}}$$

where

$$\begin{aligned} c &= \left(\frac{\pi^6\ln(2+\sqrt{3})}{2\cdot 3^7}\right)^{1/4}\frac{1}{\Gamma(1/4)}\prod_{p\equiv 1\,\mathrm{mod}\,12}\left(1-\frac{1}{p^2}\right)^{1/2}=0.2015440949... \\ &= (2.4185291388...)/12 \end{aligned}$$

and Ng [5] proved that

$$M_{1,23}(x)\sim\frac{d\,x}{(\ln x)^{1/2}}$$

where

$$d=\left\{\frac{3\sqrt{23}}{22}\prod_{p\in S}\left(1-\frac{1}{p^2}\right)^{-1}\cdot\prod_{p\in T}\left(\frac{1-1/p^2}{1-1/p^3}\right)^2\right\}^{1/2}.$$

The set S is defined as the set of all primes p with the property that the cubic polynomial y^3-y-1 has a single zero modulo p. This turns out to be the same as requiring that the Legendre symbol $(-23/p)$ be equal to -1. The set T is the set of all primes p with the property that y^3-y-1 has no zeroes modulo p (that is, it is irreducible over $\mathbb{Z}_p$). No equivalent condition involving the Legendre symbol is known [6].

It is also proved that $M_{6,6,6,6}(x)$, $M_{4,4,4,4,4,4}(x)$ and $M_{3,3,3,3,3,3,3,3}(x)$ correspond to lacunary eta products; further, each is asymptotically $C\,x/\ln(x)^{1/2}$ for some constant C. In particular, $\eta(6z)^4$ is related to the L-series for the elliptic curve $36A1$:

$$v^2=u^3+1$$

and thus it would be good to better understand the corresponding C.

By contrast, $\eta(2z)^{12}$, $\eta(z)^{24}$ and $\eta(z)^2\eta(11z)^2$ are *not* lacunary. It is conjectured that

$$M_{\underbrace{2,2,\ldots,2}_{12\text{ times}}}(x) \sim x, \qquad M_{\underbrace{1,1,\ldots,1}_{24\text{ times}}}(x) \sim x,$$

and that

$$M_{1,1,11,11}(x) \sim \left(\frac{14}{15}\prod_{a_p=0}\left(1-\frac{1}{p+1}\right)\right)x = (0.84652...)x.$$

In particular, $\eta(z)^2\eta(11z)^2$ is related to the L-series for the elliptic curve $11A3$:

$$v^2 + v = u^3 - u^2$$

and thus it would be good to compute the associated constant to higher precision.

We mention that the primes p satisfying $a_p = 0$ (as above) are called **supersingular primes**. This sequence of primes begins as 19, 29, 199, 569, 809, ... No explicit formula for a_p as a function of p, or for the nth supersingular prime, is known [7, 8, 9, 10, 11].

Another related constant for $11A3$ is

$$\gamma_j = \lim_{x\to\infty}\frac{\#\{p \le x : a_p = j\}}{\sqrt{x}/\ln(x)}$$

for any integer j. If the Lang-Trotter conjecture were proved [7], then it would follow that $\gamma_0 = 23\pi/55 \approx 1.31375$,

$$\gamma_{-1} = \frac{1}{\pi}\frac{11^2}{2^3\cdot 3^2}A \approx 0.49919, \qquad \gamma_{-2} = \frac{1}{\pi}\frac{7\cdot 11\cdot 31}{2^4\cdot 3^2\cdot 5}A \approx 0.98478$$

where

$$\begin{aligned} A &= \prod_{p\neq 2,5,11}\frac{p(p^2-p-1)}{(p-1)(p^2-1)} = \prod_{p\neq 2,5,11}\left(1-\frac{1}{(p-1)(p^2-1)}\right) \\ &= 0.9331892646... \end{aligned}$$

Some doubt exists, however, whether assumptions underlying Lang-Trotter are justified. We refer the interested reader to [12], which is a work-in-progress addressed to both mathematicians and statisticians. See also [13] for a constant, similar to A, which arises in the study of the reduced totient or Carmichael function.

A recent preprint [14] is concerned not with the density of nonzero coefficients a_k, but instead with the asymptotic mean square of a_k (which perhaps is less difficult).

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