Distribution of Error Terms

Steven Finch

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Let ν_N be a random integer chosen uniformly in $[1, N]$. Let $\varphi(n)$ denote the number of positive integers $m \leq n$ satisfying $\gcd(m, n) = 1$ and $\sigma(n)$ denote the sum of all divisors of n. The limiting probability distributions of $\varphi(\nu_N)/\nu_N$ and $\sigma(\nu_N)/\nu_N$, as $N \to \infty$, are continuous but singular in the sense that

$$F_\varphi(x) = \lim_{N\to\infty} \frac{\#\{n \leq N : \varphi(n)/n \leq x\}}{N}, \qquad F_\sigma(x) = \lim_{N\to\infty} \frac{\#\{n \leq N : \sigma(n)/n \leq x\}}{N}$$

satisfy $F'_\varphi = 0 = F'_\sigma$ almost everywhere [1, 2, 3]. Considerable effort is needed, for example, to compute that $1 - F_\sigma(2) = 0.247...$, the density of abundant numbers relative to the set of positive integers [4]. See [5, 6, 7, 8, 9] for recent work concerning F_φ and F_σ.

Starting from

$$\lim_{N\to\infty} \mathrm{E}\left(\frac{\varphi(\nu_N)}{\nu_N}\right) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{\varphi(n)}{n} = \frac{6}{\pi^2},$$

$$\lim_{N\to\infty} \mathrm{E}\left(\frac{\sigma(\nu_N)}{\nu_N}\right) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}$$

we examine distributions that perhaps are more open to analysis. Define error terms

$$H(n) = \sum_{m\leq n} \frac{\varphi(m)}{m} - \frac{6}{\pi^2}n,$$

$$K(n) = \sum_{m\leq n} \frac{\sigma(m)}{m} - \frac{\pi^2}{6}n + \frac{1}{2}\ln(n) + \frac{\gamma + \ln(2\pi)}{2}$$

then it can be shown that [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]

$$\lim_{N\to\infty} \mathrm{E}\left(H(\nu_N)\right) = \frac{3}{\pi^2}, \qquad \lim_{N\to\infty} \mathrm{Var}\left(H(\nu_N)\right) = \frac{1}{2\pi^2} - \frac{3}{\pi^4},$$

$$\lim_{N\to\infty} \mathrm{E}\left(K(\nu_N)\right) = \frac{\pi^2}{12}, \qquad \lim_{N\to\infty} \mathrm{Var}\left(K(\nu_N)\right) = \frac{5\pi^2}{144} - \frac{\pi^4}{432}.$$

Further, it is known that the limiting distributions corresponding to $H(\nu_N) - 3/\pi^2$ and $K(\nu_N) - \pi^2/12$ are symmetric and all corresponding odd moments vanish. In particular, the skewness coefficients of both quantities are zero. What is *not* precisely known are the kurtosis excesses:

$$\frac{\mathrm{E}\left[\left(H(\nu_N) - \mathrm{E}\left(H(\nu_N)\right)\right)^4\right]}{\mathrm{Var}\left(H(\nu_N)\right)^2} - 3 = -0.93...$$

$$\frac{\mathrm{E}\left[\left(K(\nu_N) - \mathrm{E}\left(K(\nu_N)\right)\right)^4\right]}{\mathrm{Var}\left(K(\nu_N)\right)^2} - 3 = 0.10...$$

which would imply that tails are thin for $H(\nu_N)$ and tails are fat for $K(\nu_N)$. This may be a consequence of the simple fact that the support of the distribution for $\varphi(\nu_N)/\nu_N$ is $[0,1]$ whereas the support of the distribution for $\sigma(\nu_N)/\nu_N$ is $[0,\infty)$.

Exact formulas for all even moments would allow us to accurately construct the distributions corresponding to $H(\nu_N)$ and $K(\nu_N)$. Evaluating the fourth moments, however, seems to be hard. Related material includes [20, 21, 22, 23].

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