

Convex Lattice Polygons

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Let $n \geq 3$ be an integer. A convex lattice n -gon is a polygon whose n vertices are points on the integer lattice \mathbb{Z}^2 and whose interior angles are strictly less than π . Let a_n denote the least possible area enclosed by a convex lattice n -gon, then [1, 2, 3]

$$\{a_n\}_{n=3}^\infty = \left\{ \frac{1}{2}, 1, \frac{5}{2}, 3, \frac{13}{2}, 7, \frac{21}{2}, 14, x, 24, \frac{65}{2}, 40, y, 59, z, 87, w, 121, \dots \right\}$$

where the unknown values x , y , z , and w are known to satisfy

$$x \in \left\{ \frac{39}{2}, \frac{41}{2}, \frac{43}{2} \right\}, \quad y \in \left\{ \frac{99}{2}, \frac{101}{2}, \frac{103}{2} \right\},$$
$$z \in \left\{ \frac{147}{2}, \frac{149}{2}, \frac{151}{2} \right\}, \quad w \in \left\{ \frac{209}{2}, \frac{211}{2}, \frac{213}{2} \right\}.$$

On the one hand, Rabinowitz [4] and Colburn & Simpson [5] demonstrated that $a_n \leq Cn^3$ for some constant $C > 0$; Zunic [6] later proved that $C \leq 1/54$. On the other hand, Andrews [7] and Arnold [8] were the first to show that $a_n \geq cn^3$ for some $c > 0$; other proofs appear in [9, 10, 11, 12]. Bárány & Tokushige [13] succeeded in proving that $\lim_{n \rightarrow \infty} a_n/n^3$ actually exists and computed that

$$\lim_{n \rightarrow \infty} \frac{a_n}{n^3} = 0.0185067\dots < \frac{1}{54}$$

via a heuristic solution of $\approx 10^{10}$ constrained minimization problems. Further, the shape of the minimizing n -gon is approximated by that of the ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

where $A = (0.003573\dots)n^2$ and $B = (1.656\dots)n$.

Much less can be said about the higher dimensional analog. A d -dimensional convex lattice polytope with n vertices has volume v_n satisfying [7, 9, 14, 15]

$$v_n \geq c_d n^{\frac{d+1}{d-1}}$$

but little else is known.

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0.1. Integer Convex Hulls. Before discussing integer convex hulls, let us mention ordinary convex hulls. Given n points chosen at random in the unit disk D , the convex hull C_n is the intersection of all convex sets containing all n points. The boundary of C_n is a polygon; let N_n denote the number of vertices of the polygon. It can be proved that [16, 17, 18]

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(N_n)}{n^{1/3}} = 2\pi\xi, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(N_n)}{n^{1/3}} = 2\pi\eta$$

where

$$\xi = \left(\frac{3\pi}{2}\right)^{-\frac{1}{3}} \Gamma\left(\frac{5}{3}\right) = 0.5384576135\dots,$$

$$\eta = \frac{16\pi^2\Gamma\left(\frac{2}{3}\right)^{-3} - 57}{27}\xi = 0.1316029298\dots = 2(0.3350302716\dots) - \xi.$$

We point out that this is more complicated than the corresponding result when the unit disk is replaced by the unit square [16, 17, 19]:

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\tilde{N}_n)}{\ln(n)} = \frac{8}{3}, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(\tilde{N}_n)}{\ln(n)} = \frac{40}{27}.$$

In the integer case, we consider not n random points in D , but rather *all* lattice points in rD , the disk of radius r , where r is large. The convex hull C_r of all these lattice points is clearly a convex lattice polygon, together with its interior. Motivation for studying this polygon comes from integer programming: When maximizing a linear function φ on the lattice points in rD (or any given convex set in \mathbb{R}^2), one looks for the maximum point of φ on C_r . The size of the programming problem is hence proportional to N_r , the number of vertices of C_r , and thus we wish to have bounds on N_r .

Balog & Bárány [20, 21] proved that, for sufficiently large r ,

$$0.33r^{2/3} \leq N_r \leq 5.54r^{2/3}$$

but confessed that it isn't clear whether $\lim_{r \rightarrow \infty} N_r r^{-2/3}$ exists. It is possible, however, to obtain asymptotics for the average value of N_r , defined in a special way:

$$\mathbb{E}_\theta(N_r) = \frac{1}{r^\theta} \int_r^{r+r^\theta} N_\rho d\rho$$

where the parameter θ satisfies $0 < \theta < 1$. (Actually, the only feature needed of r^θ is that it increases with r , but less rapidly than r itself.) Balog & Deshouillers [22] proved that

$$\lim_{r \rightarrow \infty} \frac{\mathbb{E}_\theta(N_r)}{r^{2/3}} = \frac{6 \cdot 2^{2/3}}{\pi} \chi = 3.4536898915\dots$$

independently of θ , where χ is defined later. The growth rate $2/3$ is what we would expect on the basis of the probabilistic model (ordinary convex hull case), but the preceding constant $3.453\dots$ is slightly different from $2\pi\xi = 3.383\dots$. In this sense, lattice points do not behave in the same way as random points.

Another occurrence of the constant χ is as follows. For real x , let $\|x\|$ denote the distance from x to the nearest integer. Then, for $0 \leq a < b \leq 1$, we have [22]

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{(b-a)\lambda^{1/3}} \int_a^b \min_{t \neq 0} (|\alpha t| + \lambda t^2) d\alpha = \frac{6}{\pi^2} \chi.$$

If $\lambda = 0$, the integral clearly is zero since, for any α , the point $t = 1/\alpha$ gives the minimum. If $\lambda > 0$, this strategy no longer works because the penalty term $\lambda t^2 = \lambda/\alpha^2$ would be large.

Let Δ denote the triangular region bounded by the lines $y = x$, $y = 1 - x$ and $x = 1$. Partition Δ into four domains:

$$\begin{aligned} \Delta_1 &= \{(x, y) \in \Delta : 1 \leq xy(x+y)\}, \\ \Delta_2 &= \{(x, y) \in \Delta : xy(x+y) \leq 1 \leq x(x+y)(x+2y)\}, \\ \Delta_3 &= \{(x, y) \in \Delta : x(x+y)(x+2y) \leq 1 \leq x(x+y)(2x+y)\}, \\ \Delta_4 &= \{(x, y) \in \Delta : x(x+y)(2x+y) \leq 1\}. \end{aligned}$$

Define $F : \Delta \rightarrow \mathbb{R}$ by

$$F(x, y) = \begin{cases} 4 - x^3 - y^3 & \text{in } \Delta_1, \\ \frac{1}{xy(x+y)} + 2 - (x+y)(x-y)^2 & \text{in } \Delta_2, \\ \frac{1}{y(x+y)(x+2y)} + 6 - (x+y)(3x^2 + 2xy + y^2) & \text{in } \Delta_3, \\ \frac{1}{x(x+y)(2x+y)} + \frac{1}{y(x+y)(x+2y)} + 4 - (x+y)(x^2 + xy + y^2) & \text{in } \Delta_4, \end{cases}$$

then χ is given by

$$\chi = \int_{1/2}^1 \int_{1-x}^x F(x, y) dy dx.$$

Again, much less can be said about the higher dimensional analog. Let B_d denote the d -dimensional unit ball. The number of vertices, N_r , of the integer convex hull of rB_d satisfies [23]

$$c_d r^{\frac{d(d-1)}{d+1}} \leq N_r \leq C_d r^{\frac{d(d-1)}{d+1}}$$

but an asymptotic average value for N_r is not known for any $d \geq 3$.

0.2. Addendum. The d -dimensional unit cube has 2^d vertices. Randomly select $n = n(d)$ vertices with replacement and form the ordinary convex hull of these points. If V_d denotes its expected volume, then for any $\varepsilon > 0$, [24, 25]

$$\lim_{d \rightarrow \infty} V_d = \begin{cases} 0 & \text{if } n(d) \leq (2/\sqrt{e} - \varepsilon)^d, \\ 1 & \text{if } n(d) \geq (2/\sqrt{e} + \varepsilon)^d. \end{cases}$$

This is an interesting occurrence of the constant $2/\sqrt{e} = 1.2130613194\dots$, which is surprisingly small (relative to 2)! If instead the n points are selected uniformly in the interior of the d -cube, then the same threshold phenomenon occurs, with constant $2/\sqrt{e}$ replaced by

$$\exp \left(\int_0^\infty \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)^2 dx \right) = 2.1396909474\dots$$

In fact, a closed-form expression is possible since

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)^2 dx = \ln(2\pi) - \gamma - \frac{1}{2} = 0.7606614015\dots$$

and the details underlying this formula appear in [26]. See [25] for relevant discussion of the d -dimensional unit ball.

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