Coefficient Estimates for Univalent Functions

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A complex analytic function f defined on the open unit disk D is **univalent** (or **schlicht**) if f is one-to-one; that is, $f(z) = f(w)$ if and only if $z = w$. We are interested in estimating the coefficients $\{a_n\}_{n=0}^{\infty}$ of the Maclaurin series expansion $\sum_{n=0}^{\infty} a_n z^n$ of $f(z)$. Define a set

$$S = \{f : D \to \mathbb{C} : f \text{ is univalent, } f(0) = 0 \text{ and } f'(0) = 1\}$$

and subsets

$$S_{\mathbb{R}} = \{f \in S : a_n \in \mathbb{R} \text{ for all } n \geq 2\},$$
$$S_{\text{odd}} = \{f \in S : f(z) = -f(-z) \text{ for all } z \in D\},$$
$$S_M = \{f \in S : |f(z)| < M \text{ for all } z \in D\},$$

where $M > 1$. On the one hand, the Koebe function

$$\kappa(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n\, z^n$$

is a member of $S_{\mathbb{R}}$ but not of $S_{\text{odd}} \cup S_M$. On the other hand, the Pick function

$$P_M(z) = M\, \kappa^{-1}\left(\frac{\kappa(z)}{M}\right), \qquad \text{where} \quad \kappa^{-1}(w) = \frac{2w + 1 - \sqrt{4w+1}}{2w},$$

is a member of $S_{\mathbb{R}} \cap S_M$ but not of S_{odd}. De Branges [1, 2] proved Bieberbach's famous conjecture [3]:

$$\max_{f \in S} |a_n| = n$$

which occurs if and only if f is a rotation of κ; equivalently, $f(z) = e^{-i\theta}\kappa(e^{i\theta})$ for some $\theta \in \mathbb{R}$. Actually, he proved something even more subtle: Milin's conjecture, which involves not the coefficients $\{a_n\}$ but rather the **logarithmic coefficients** $\{b_n\}$, where

$$\ln\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} b_n z^n.$$

It is surprising how much material here remains unresolved, even twenty years after de Branges' achievement!

0.1. Bombieri's Conjecture. While proving a local version of Bieberbach's conjecture, Bombieri [4] speculated about a formula for

$$\sigma_{m,n} = \liminf_{\substack{f \to \kappa \\ f \in S}} \frac{n - \mathrm{Re}(a_n)}{m - \mathrm{Re}(a_m)}$$

where $m \geq 2$, $n \geq 2$ and where $f \to \kappa$ means locally uniform convergence on D (uniform on every compact subset of D). He determined, for example, that

$$\liminf_{\substack{f \to \kappa \\ f \in S}} \frac{3 - \mathrm{Re}(a_3)}{\left(2 - \mathrm{Re}(a_2)\right)^{3/2}} = \frac{8}{3}$$

and hence $\sigma_{2,3} = 0$. Likewise, $\sigma_{4,3} = 0$. Bshouty & Hengartner [5] proved Bombieri's conjecture for f with real coefficients:

$$\liminf_{\substack{f \to \kappa \\ f \in S_{\mathbb{R}}}} \frac{n - a_n}{m - a_m} = \min_{0 \leq \theta < 2\pi} \frac{n \sin(\theta) - \sin(n\theta)}{m \sin(\theta) - \sin(m\theta)} = \beta_{m,n}$$

but the case of f with complex coefficients was left open. The first counterexample to Bombieri's conjecture was found by Greiner & Roth [6]:

$$\sigma_{3,2} = \frac{e-1}{4e} = 0.1580301397... < 0.25 = \beta_{3,2} = \frac{1}{4}.$$

Prokhorov & Vasil'ev [7] gave additional counterexamples:

$$\sigma_{4,2} = 0.050057... < 0.1 = 1/10 = \beta_{4,2},$$

$$\sigma_{2,4} = 0.969556... < 1 = \beta_{2,4},$$

$$\sigma_{3,4} = 0.791557... < 0.828427... < 2\left(\sqrt{2} - 1\right) = \beta_{3,4}.$$

Their interesting work involves Löwner's differential equation, Pontryagin's maximum principle and the numerical solution of an optimal control system.

0.2. Fekete-Szegö Theorem. Littlewood & Paley [8] proved that the coefficients in S_{odd} are bounded; that is, there exists $A > 0$ for which $|a_{2n+1}| \leq A$ for all $f \in S_{\mathrm{odd}}$ and all $n \geq 1$. In a footnote to their paper, they wrote "No doubt the true bound is given by $A = 1$." It is clearly true that $\max_{f \in S_{\mathrm{odd}}} |a_3| = 1$. Fekete & Szegö [9, 10], however, disproved the Littlewood-Paley conjecture for the next coefficient:

$$\alpha = \max_{f \in S_{\mathrm{odd}}} |a_5| = \frac{1}{2} + e^{-2/3} = 1.0134171190...$$

Schaeffer & Spencer [11] exhibited explicitly the unique extremal function f and noted that $f \in S_{\mathbb{R}}$ as well. They demonstrated that

$$\max_{f \in S_{\mathrm{odd}} \cap S_{\mathbb{R}}} |a_{2n+1}| > 1$$

for each $n \geq 2$. Leeman [12] studied the case $n = 3$:

$$\max_{f \in S_{\mathrm{odd}} \cap S_{\mathbb{R}}} |a_7| = \frac{1090}{1083} = 1.0064635272...$$

and such extremal functions f must additionally satisfy $a_3 = \pm 18/19$ and $a_5 = 351/261$. The occurrence of rational numbers here is quite surprising. The best general estimate is due to Hu Ke [13], improving upon [8, 14, 15, 16, 17, 18]:

$$\max_{f \in S_{\mathrm{odd}}} |a_{2n+1}| \leq 1.1305...$$

Ke's proof is based on Milin's conjecture (now de Branges' theorem), which we will discuss shortly.

0.3. Tammi's Conjecture.

The following estimates hold for the bounded univalent function scenario:

$$\max_{f \in S_M} |a_2| = 2\left(1 - M^{-1}\right),$$

$$\max_{f \in S_M} |a_3| = \begin{cases} 1 - M^{-2} & \text{if } 1 < M < e, \\ 1 - M^{-2} + 2\left(\lambda - M^{-1}\right)^2 & \text{if } M \geq e, \end{cases}$$

$$\max_{f \in S_M} |a_4| = \begin{cases} \frac{2}{3}\left(1 - M^{-3}\right) & \text{if } 1 < M \leq \frac{34}{19}, \\ 2\left(2 - 10M^{-1} + 15M^{-2} - 7M^{-3}\right) & \text{if } M \geq \mu, \end{cases}$$

where the parameter λ is the largest of the two real solutions of $\lambda \ln(\lambda) + M^{-1} = 0$ and the constant μ is the smallest for which the formula holds (to be ascertained). The first estimate dates back to Pick [19]; the second is due to Löwner [20], Schaeffer & Spencer [21] and Janowski [22]; and the third comes from Schiffer & Tammi [23], who computed that $\mu \leq 100/3$. It turns out, for large M, that $\max_{f \in S_M} |a_{2n}|$ is the $(2n)^{\mathrm{th}}$ coefficient in the Maclaurin series expansion of the Pick function $P_M(z)$, for any $n \geq 1$ [24, 25].

Note the sizable gap in the formula for $\max_{f \in S_M} |a_4|$. Tammi [26] determined, when f has real coefficients and $M \geq 11$, that

$$\max_{f \in S_{\mathbb{R}} \cap S_M} |a_4| = 2\left(2 - 10M^{-1} + 15M^{-2} - 7M^{-3}\right).$$

The formula fails for $M < 11$. Hence it was natural for him to conjecture [27] that $\mu = 11$ for f with complex coefficients as well. Prokhorov & Vasil'ev [7] disproved this conjecture, showing that $\mu = 22.9569...$, again using a numerical optimal control-based approach.

0.4. Greiner-Roth Theorem. Elaborate expressions built from series coefficients can also be optimized. Greiner & Roth [28], starting from [29, 30], proved that the function $f \in S$ maximizing

$$\operatorname{Re}\left(a_3 + \frac{p-3}{3}a_2^2\right) + \frac{p+1}{3}|a_2|^2, \qquad p \in \mathbb{R} \text{ fixed},$$

is

$$f(z) = \begin{cases} \pm i\,K(\mp i\,z) & \text{if } p \le \dfrac{3}{4\ln(2)} - \dfrac{1}{2} = 0.5820212806..., \\ \pm K(\pm z) & \text{if } p \ge \dfrac{1}{2}\dfrac{2e^3+1}{e^3-1} = 1.0785935447.... \end{cases}$$

In the gap, f cannot be a rotation of the Koebe function. Starting from [31], they also proved that the function $f \in S$ maximizing

$$\operatorname{Re}\left(a_3 - q\,a_2^2\right) + q|a_2|^2, \qquad q \in \mathbb{R} \text{ fixed},$$

is

$$f(z) = \begin{cases} \pm K(\pm z) & \text{if } q \le \dfrac{1}{2} = 0.5, \\ \pm i\,K(\mp i\,z) & \text{if } q \ge \dfrac{1}{2}\dfrac{e}{e-1} = 0.7909883534.... \end{cases}$$

Again, in the gap, f can be proved not to be a rotation of the Koebe function. Such expressions serve to generalize those used to obtain the Fekete-Szegö constant α mentioned earlier. Explicit formulas for the gap extremals are not available, but these functions can be found numerically via optimal control.

0.5. Milin's Constant. Define

$$\delta = \sup_{n\ge1}\sup_{f\in S}\sum_{k=1}^{n}\left(k|b_k|^2 - \frac{1}{k}\right),$$

then it can be shown [10, 16, 17] that $0.0266 < 2\ln(\alpha) < \delta < 0.3119 < \operatorname{Ei}(\ln(2))/2 - \gamma - \ln(\ln(2))$, where Ei is the exponential integral and γ is Euler's constant. A more precise estimate of Milin's constant δ would be good to see, as well as the corresponding extremal functions. Note that, if $f = \kappa$, then the logarithmic coefficients $b_n = 1/n$ for all n; hence the Koebe function is far from optimal in this setting. It is known that $\max_{f\in S_{\text{odd}}}|a_{2n+1}| < e^{\delta/2}$ (which gave, at one time, the best general estimate 1.17 of the odd coefficents); if it were true that $\delta = 0$, then the Littlewood-Paley conjecture would follow.

In contrast, we have

$$\hat{\delta} = \sup_{n\ge1}\sup_{f\in S}\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|b_k|^2 - \frac{1}{k}\right) = 0,$$

which is Milin's conjecture (now proved, as stated earlier). Here, of course, the Koebe function is optimal. For $f \neq \kappa$, the lower order contributions to the sum evidently tend to be negative, forcing $\hat{\delta} < \delta$. See also [32, 33].

0.6. Bieberbach-Eilbenberg Functions.

Define a new set

$$\tilde{S} = \{f : D \to \mathbb{C} : f \text{ is univalent, } f(0) = 0 \text{ and } f(z)f(w) \neq 1 \text{ for any } z, w \in D\}.$$

Note that nothing is assumed about a_1. In fact,

$$\max_{f \in \tilde{S}} |a_1| = 1$$

which occurs if and only if $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Nehari [34] and Aharonov [35] proved that

$$\frac{e^{-1/2}}{\sqrt{n}} \leq \max_{f \in \tilde{S}} |a_n| < \frac{e^{-\gamma/2}}{\sqrt{n-1}}$$

for all $n \geq 2$; in particular, $|a_2|$ is less than $e^{-\gamma/2} < 0.74931$. Hummel & Schiffer [36, 37] obtained the estimate

$$\max_{f \in \tilde{S}} |a_2| = \frac{1}{2}\eta = 0.5811002808...$$

where $\eta = 1.1622005617...$ is the unique real solution of the equation

$$\int_0^1 \left(\frac{1-t}{\eta^2 + t^2}\right)^{1/2} dt = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \left[\left(1 + \eta^2 \sin(\theta)^2\right)^{1/2} - 1\right]^{1/2} d\theta.$$

Another interesting result is the estimate

$$\max_{f \in \tilde{S}} |a_1 a_2| \leq \frac{8}{27}\eta^2 = 0.4002104135...$$

and we wonder about higher order coefficients of such functions.

0.7. Krzyz's Conjecture.

Define two new sets

$$U = \{f : D \to \mathbb{C} : f \text{ is analytic, } 0 < |f(z)| < 1\}, \qquad V = \{f \in U : f \text{ is univalent}\}.$$

Obviously $U \cap S = \emptyset$ and $0 < |a_0| < 1$ for every $f \in U$. For $n \geq 1$, Krzyz [38] conjectured that

$$\max_{f \in U} |a_n| = \frac{2}{e} = 0.7357588823...$$

which occurs if and only if $f(z)=e^{(z^n+1)/(z^n-1)}$ or a rotation of this. Note that f is not univalent. Krzyz's conjecture has been proved only for $n\leq 5$ [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]. A general estimate also applies [51, 52, 53]:

$$\max_{f\in U}|a_n|<0.99918...$$

For univalent functions, Prokhorov & Szynal [37] demonstrated that

$$\max_{f\in V}|a_1|=12-8\sqrt{2}=0.6862915010...,$$

$$\max_{f\in V}|a_2|=\frac{8\xi(1-\xi)(1-2\xi-\xi^2)}{(1+\xi)^3}=0.4553841384...$$

where $\xi=0.1414780159...$ has minimal polynomial $\xi^4+4\xi^3+6\xi^2-8\xi+1$.

We close with one more problem. Grinshpan [54, 55], improving upon [56, 57, 58], showed that $-2.97<|a_{n+1}|-|a_n|<3.61$ for all $f\in S$ and all $n\geq 1$. It is further known that the constants on the left and right cannot be replaced by -1 and 1, respectively, even if we restrict discussion to $f\in S_{\text{odd}}$ [59]. See other related problems in [60, 61].

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