

# Class Number Theory

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The problem of representing an integer as a sum of squares, or more generally as the value of a quadratic form, is very old and challenging [1, 2, 3, 4, 5, 6, 7]. We will barely scratch the surface of this enormous literature.

**0.1. Form Class Group.** A binary quadratic form  $f(x, y) = ax^2 + bxy + cy^2$  with  $a, b, c \in \mathbb{Z}$  is **primitive** if  $a, b, c$  are relatively prime and has **discriminant**  $\delta_f = b^2 - 4ac$ . The form  $f$  is **positive definite** if the matrix

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

is positive definite (meaning  $a > 0$  and  $\delta_f < 0$ ) and **indefinite** if  $\delta_f > 0$ . An integer  $d$  is a discriminant  $\delta_f$  for some form  $f$  if and only if  $d \equiv 0, 1 \pmod{4}$ . A discriminant  $D \neq 0, 1$  is a **fundamental discriminant** assuming that

$$D = \begin{cases} m & \text{if } m \equiv 1 \pmod{4}, \\ 4m & \text{if } m \equiv 2, 3 \pmod{4} \end{cases}$$

for some square-free integer  $m$ . Every nonsquare discriminant  $d$  can be uniquely expressed as  $De^2$  where  $D$  is a fundamental discriminant and  $e \geq 1$ . A partial listing of fundamental discriminants appears in Table 1 and the correspondence  $m \leftrightarrow D$  will be needed later [8].

Table 1 *Interplay between  $m$  and  $D$ ,  $-163 \leq D \leq 136$*

$m$	-3	-1	-7	-2	-11	-15	-19	-5	-23	-6	-31	-35	...	-163
$D$	-3	-4	-7	-8	-11	-15	-19	-20	-23	-24	-31	-35	...	-163
$m$	5	2	3	13	17	21	6	7	29	33	37	10	...	34
$D$	5	8	12	13	17	21	24	28	29	33	37	40	...	136

Assume that  $D$  is a fundamental discriminant. Two quadratic forms  $f, g$  with  $\delta_f = D = \delta_g$  are **properly equivalent** if there is a linear change of variables

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad ru - st = 1, \quad r, s, t, u \in \mathbb{Z}$$

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for which  $f(x, y) = g(x', y')$  always. We say that  $f, g$  are in the same **form class** and define the **form class number**

$$h^+(D) = \begin{cases} \text{the number of classes of primitive positive} & \text{if } D < 0, \\ \text{definite forms of discriminant } D & \\ \text{the number of classes of primitive} & \text{if } D > 0. \\ \text{indefinite forms of discriminant } D & \end{cases}$$

For example,  $h^+(-4) = 1$  and  $x^2 + y^2$  is a representative element of the unique form class of discriminant  $-4$ ;  $h^+(-20) = 2$  and  $x^2 + 5y^2$ ,  $2x^2 + 2xy + 3y^2$  are representative elements of the two corresponding classes of discriminant  $-20$ .

It is possible to endow the set of form classes, for fixed  $D$ , with the structure of an abelian group. We simply illustrate in the case  $D = -4$ :

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = x_3^2 + y_3^2$$

where

$$x_3 = x_1x_2 - y_1y_2, \quad y_3 = x_1y_2 + y_1x_2;$$

and in the case  $D = -20$ :

$$\begin{aligned} (x_1^2 + 5y_1^2)(x_2^2 + 5y_2^2) &= x_4^2 + 5y_4^2, \\ (x_1^2 + 5y_1^2)(2x_2^2 + 2x_2y_2 + 3y_2^2) &= 2x_5^2 + 2x_5y_5 + 3y_5^2, \\ (2x_1^2 + 2x_1y_1 + 3y_1^2)(2x_2^2 + 2x_2y_2 + 3y_2^2) &= x_6^2 + 5y_6^2 \end{aligned}$$

where

$$\begin{aligned} x_4 &= x_1x_2 - 5y_1y_2, & y_4 &= x_1y_2 + y_1x_2, \\ x_5 &= x_1x_2 - y_1x_2 - 3y_1y_2, & y_5 &= x_1y_2 + 2y_1x_2 + y_1y_2, \\ x_6 &= 2x_1x_2 + x_1y_2 + y_1x_2 - 2y_1y_2, & y_6 &= x_1y_2 + y_1x_2 + y_1y_2. \end{aligned}$$

This multiplication is called **Gaussian composition** and is perhaps best understood via the following section.

We discuss two variations of the preceding. If the determinant of the linear transformation  $(x, y) \mapsto (x', y')$  is allowed to be  $ru - st = \pm 1$ , then the corresponding number of equivalence classes is [9]

$$\hat{h}(D) = \frac{1}{2} \left( h^+(D) + 2^{\omega(D)-1} \right)$$

where  $\omega(n)$  denotes the number of distinct prime factors of  $|n|$ . Rephrasing,  $h^+(D)$  is the number of orbits under the action of the matrix group  $\text{SL}_2(\mathbb{Z})$  on the primitive

binary quadratic forms of discriminant  $D$ , while  $\hat{h}(D)$  is the same under the action of  $\text{GL}_2(\mathbb{Z})$ . For instance,  $h^+(-23) = 3 > 2 = \hat{h}(-23)$  and  $h^+(136) = 4 > 3 = \hat{h}(136)$ .

The second variation seems quite artificial but is actually important. Two quadratic forms  $f, g$  with  $\delta_f = D = \delta_g$  are **vulgarily equivalent** if there is a linear change of variables

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad ru - st = \theta = \pm 1, \quad r, s, t, u \in \mathbb{Z}$$

for which  $f(x, y) = \theta g(x', y')$  always. Note the factor  $\theta$  in front of  $g$ . Define  $h(D)$  to be the number of vulgar equivalence classes of primitive quadratic forms of discriminant  $D$ . Note here that forms are not assumed to be positive definite for  $D < 0$ . As an example,  $h^+(12) = 2 > 1 = h(12)$  since the forms  $-3x^2 + y^2$  and  $-x^2 + 3y^2$  are not properly equivalent, but are vulgarily equivalent via the assignment  $(x', y') = (y, x)$ .

**0.2. Ideal Class Group.** Let  $m \neq 0, 1$  be a square-free integer. The **quadratic number field**

$$\mathbb{Q}(\sqrt{m}) = \mathbb{Q} + \mathbb{Q}\sqrt{m} = \{u + v\sqrt{m} : u, v \in \mathbb{Q}\}$$

is the smallest subfield of  $\mathbb{C}$  containing both  $\mathbb{Q}$  and  $\sqrt{m}$ . An element  $\alpha \in \mathbb{Q}(\sqrt{m})$  is an **algebraic integer** if it is a zero of a monic polynomial  $z^2 + bz + c$  with  $b, c \in \mathbb{Z}$ . The set of algebraic integers of  $\mathbb{Q}(\sqrt{m})$  is the subring

$$\mathcal{O}_m = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{m} & \text{if } m \equiv 2, 3 \pmod{4}, \\ \mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{m}}{2} & \text{if } m \equiv 1 \pmod{4} \end{cases}$$

of  $\mathbb{Q}(\sqrt{m})$ , often called the **maximal order** or simply the **integers**. Using the correspondence between the **radicand**  $m$  and the fundamental discriminant  $D$ , we have

$$\mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{D}), \quad \mathcal{O}_m = \mathbb{Z} + \mathbb{Z}\frac{D+\sqrt{D}}{2}.$$

For example,  $\mathcal{O}_{-1}$  is the ring of Gaussian integers. In  $\mathcal{O}_{-5}$ , we have a surprising failure of unique factorization:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

More will be said about this momentarily.

An **ideal**  $I$  of  $\mathcal{O}_m$  is an additive subgroup of  $\mathcal{O}_m$  with the property that, if  $\alpha \in I$  and  $\rho \in \mathcal{O}_m$ , then  $\rho\alpha \in I$ . The set

$$(\alpha) = \{\rho\alpha : \rho \in \mathcal{O}_m\}$$

is the ideal of all multiples of a single element  $\alpha \in \mathcal{O}_m$  and is called a **principal ideal**. The ideal

$$(\alpha_1, \alpha_2) = \{\rho_1\alpha_1 + \rho_2\alpha_2 : \rho_1, \rho_2 \in \mathcal{O}_m\}$$

is **nonprincipal** if  $(\alpha_1, \alpha_2) \neq (\alpha_3)$  for any  $\alpha_3 \in \mathcal{O}_m$ . The **product**  $IJ$  of two ideals is the ideal of all finite sums of products of the form  $\alpha\beta$  with  $\alpha \in I$  and  $\beta \in J$ . In  $\mathcal{O}_{-5}$ , the principal ideal (6) can be written as

$$\begin{aligned} (6) &= (2)(3) = I_1^2 I_2 I_3 \\ &= (1 + \sqrt{-5})(1 - \sqrt{-5}) = I_1 I_2 I_1 I_3 \end{aligned}$$

where

$$\begin{aligned} I_1 &= (2, 1 + \sqrt{-5}) = (2, 1 - \sqrt{-5}), \\ I_2 &= (3, 1 + \sqrt{-5}), \quad I_3 = (3, 1 - \sqrt{-5}). \end{aligned}$$

Thus the two distinct factorizations of the number 6 in  $\mathcal{O}_{-5}$  come from permuting  $I_1, I_2, I_3$  in the factorization of the ideal (6).

Given  $\alpha = u + v\sqrt{m} \in \mathbb{Q}(\sqrt{m})$ , define its **conjugate**  $\bar{\alpha} = u - v\sqrt{m}$  and its **norm**  $N(\alpha) = \alpha\bar{\alpha} = u^2 - mv^2$ . If  $\alpha \in \mathcal{O}_m$ , then clearly  $\bar{\alpha} \in \mathcal{O}_m$  and  $N(\alpha) \in \mathbb{Z}$ . Given an ideal  $I$  of  $\mathcal{O}_m$ , define its conjugate  $\bar{I} = \{\bar{\alpha} : \alpha \in I\}$  and its norm  $N(I) = \gcd\{N(\alpha) : \alpha \in I\}$ . For example, if  $I$  is the principal ideal  $(\alpha)$ , then  $\bar{I} = (\bar{\alpha})$  and  $N(I) = |N(\alpha)|$ . If  $I$  and  $J$  are two ideals, then  $N(IJ) = N(I)N(J)$ ; also  $I\bar{I} = (N(I))$  is principal.

Two ideals  $I, J$  of  $\mathcal{O}_m$  are **strictly equivalent** if there exist  $\alpha, \beta \in \mathcal{O}_m$  such that

$$(\alpha)I = (\beta)J, \quad N(\alpha\beta) > 0.$$

We say that  $I, J$  are in the same **narrow ideal class** and define  $H_m^+$  to be the finite abelian group of ideals modulo this relation. If the requirement that  $N(\alpha\beta) > 0$  is removed, we instead say that  $I, J$  are in the same **wide ideal class** and define  $H_m$  analogously.  $H_m^+$  is called the **narrow class group** and its cardinality  $h_m^+$  is the **narrow class number**. The name for  $H_m$  is often abbreviated simply to **class group**. The **class number**  $h_m$  can be found in terms of  $h_m^+$  via

$$h_m = \begin{cases} h_m^+ & \text{if } m < 0 \text{ or } (m > 0 \text{ and } N(\varepsilon) = -1), \\ \frac{1}{2}h_m^+ & \text{if } m > 0 \text{ and } N(\varepsilon) = 1 \end{cases}$$

where  $\varepsilon$  is the **fundamental unit** of  $\mathcal{O}_m$  (to be defined in the next section). Group-theoretic properties of  $H_m$  and the efficient computation of  $h_m$  have attracted much attention in recent years.

It turns out that the abelian group of classes of primitive binary quadratic forms of discriminant  $D$  is isomorphic to the narrow class group  $H_m^+$ , where the interplay  $m \leftrightarrow D$  was described earlier. In particular, Gaussian composition of forms can be elegantly written using ideals and  $h^+(D) = h_m^+$ ; see Tables 2 and 3 [10]. By the same reasoning, we have  $h(D) = h_m$  but no interpretation of  $\hat{h}(D)$  in ideal class theory

seems to be useful. Our convention for treating the discriminant  $D$  as an argument and the radicand  $m$  as a subscript is perhaps new.

Table 2 *Class Numbers as Functions of  $m$ ,  $-163 \leq m \leq 34$*

$m$	-1	-2	-3	-5	-6	-7	-10	-11	-13	-14	-15	-17	...	-163
$h_m$	1	1	1	2	2	1	2	1	2	4	2	4	...	1
$\hat{h}_m$	1	1	1	2	2	1	2	1	2	3	2	3	...	1
$m$	2	3	5	6	7	10	11	13	14	15	17	19	...	34
$h_m^+$	1	2	1	2	2	2	2	1	2	4	1	2	...	4
$h_m$	1	1	1	1	1	2	1	1	1	2	1	1	...	2
$\hat{h}_m$	1	2	1	2	2	2	2	1	2	4	1	2	...	3

Table 3 *Class Numbers as Functions of  $D$ ,  $-163 \leq D \leq 136$*

$D$	-3	-4	-7	-8	-11	-15	-19	-20	-23	-24	-31	-35	...	-163
$h(D)$	1	1	1	1	1	2	1	2	3	2	3	2	...	1
$\hat{h}(D)$	1	1	1	1	1	2	1	2	2	2	2	2	...	1
$D$	5	8	12	13	17	21	24	28	29	33	37	40	...	136
$h^+(D)$	1	1	2	1	1	2	2	2	1	2	1	2	...	4
$h(D)$	1	1	1	1	1	1	1	1	1	1	1	2	...	2
$\hat{h}(D)$	1	1	2	1	1	2	2	2	1	2	1	2	...	3

A maximal order  $\mathcal{O}_m$  is a UFD (unique factorization domain) if and only if it is a PID (principal ideal domain), which is true if and only if  $h_m = 1$ . Also,  $h_m \leq 2$  if and only if any two decompositions of  $\alpha \in \mathcal{O}_m$  into products of irreducible elements must possess the same number of factors [11, 12, 13, 14]. Hence the class number measures, in a vague sense, how far  $\mathcal{O}_m$  is from being a UFD.

**0.3. Fundamental Unit.** Let  $m > 1$  be square-free. A **unit**  $\varepsilon \in \mathcal{O}_m$  satisfies  $N(\varepsilon) = \pm 1$ ; it is the **fundamental unit** if  $\varepsilon > 1$  and every other unit is of the form  $\pm \varepsilon^n$ ,  $n \in \mathbb{Z}$ . Here is a conceptually simple algorithm for computing  $\varepsilon$ . If  $m \equiv 2, 3 \pmod 4$ , calculate  $mb^2$  for  $b = 1, 2, 3, \dots$  and stop at the first integer  $mb_0^2$  that differs from a square  $a_0^2$  by exactly  $\pm 1$ ; then  $\varepsilon = a_0 + b_0\sqrt{m}$ . If  $m \equiv 1 \pmod 4$ , stop instead at the first integer  $mb_0^2$  that differs from a square  $a_0^2$  by exactly  $\pm 4$ ; then  $\varepsilon = (a_0 + b_0\sqrt{m})/2$ . In both cases, we assume that  $a_0 \geq 1$ .

Two alternative algorithms involve continued fractions [15, 16]. For the first, define

$$\mu = \begin{cases} \frac{1+\sqrt{m}}{2} & \text{if } m \equiv 1 \pmod 4, \\ \sqrt{m} & \text{if } m \equiv 2, 3 \pmod 4 \end{cases} = \frac{P_0 + \sqrt{m}}{Q_0}$$

and let the (eventually periodic) continued fraction expansion of  $\mu$  be

$$\mu = c_0 + \frac{1|}{|c_1} + \frac{1|}{|c_2} + \frac{1|}{|c_3} + \dots$$

Define

$$P_{j+1} = c_j Q_j - P_j, \quad Q_{j+1} = \frac{m - P_{j+1}^2}{Q_j}$$

for  $j \geq 0$ , so that

$$\frac{P_j + \sqrt{m}}{Q_j} = c_j + \frac{1|}{|c_{j+1}} + \frac{1|}{|c_{j+2}} + \frac{1|}{|c_{j+3}} + \dots$$

and hence

$$\varepsilon = \prod_{j=1}^{\lambda} \frac{P_j + \sqrt{m}}{Q_j}$$

where  $\lambda$  is the period length of the continued fraction expansion for  $\mu$ .

The second possesses a curiously ambiguous outcome. Let

$$\sqrt{m} = d_0 + \frac{1|}{|d_1} + \frac{1|}{|d_2} + \frac{1|}{|d_3} + \dots$$

and define

$$\begin{aligned} A_0 &= d_0, & A_1 &= d_0 d_1 + 1, & B_0 &= 1, & B_1 &= d_1, \\ A_k &= d_k A_{k-1} + A_{k-2}, & B_k &= d_k B_{k-1} + B_{k-2}, \end{aligned}$$

for  $k \geq 2$ , so that

$$\frac{A_k}{B_k} = d_0 + \frac{1|}{|d_1} + \frac{1|}{|d_2} + \dots + \frac{1|}{|d_k} = \text{the } k^{\text{th}} \text{ convergent of } \sqrt{m}.$$

Let  $l$  denote the period length of the continued fraction expansion for  $\sqrt{m}$ . It can be proved that, if  $m \not\equiv 5 \pmod{8}$ , then  $\varepsilon = A_{l-1} + B_{l-1}\sqrt{m}$ . If  $m \equiv 5 \pmod{8}$ , however, all we can conclude is that  $A_{l-1} + B_{l-1}\sqrt{m}$  is either  $\varepsilon$  or  $\varepsilon^3$ . See Tables 4 and 5 [17].

Table 4 *Fundamental Unit  $\varepsilon$  and Norm  $N(\varepsilon)$  as Functions of  $m$ ,  $2 \leq m \leq 17$*

$m$	2	3	5	6	7	10	11	13	14	15	17
$\varepsilon$	$\frac{1+\sqrt{2}}{1}$	$\frac{2+\sqrt{3}}{1}$	$\frac{1+\sqrt{5}}{2}$	$\frac{5+2\sqrt{6}}{1}$	$\frac{8+3\sqrt{7}}{1}$	$\frac{3+\sqrt{10}}{1}$	$\frac{10+3\sqrt{11}}{1}$	$\frac{3+\sqrt{13}}{2}$	$\frac{15+4\sqrt{14}}{1}$	$\frac{4+\sqrt{15}}{1}$	$\frac{4+\sqrt{17}}{1}$
$N(\varepsilon)$	-1	+1	-1	+1	+1	-1	+1	-1	+1	+1	-1

Table 5 *Fundamental Unit  $\varepsilon$  and Norm  $N(\varepsilon)$  as Functions of  $D$ ,  $5 \leq D \leq 37$*

$D$	5	8	12	13	17	21	24	28	29	33	37
$\varepsilon$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{2}}{1}$	$\frac{2+\sqrt{3}}{1}$	$\frac{3+\sqrt{13}}{2}$	$\frac{4+\sqrt{17}}{1}$	$\frac{5+\sqrt{21}}{2}$	$\frac{5+2\sqrt{6}}{1}$	$\frac{8+3\sqrt{7}}{1}$	$\frac{5+\sqrt{29}}{2}$	$\frac{23+4\sqrt{33}}{1}$	$\frac{6+\sqrt{37}}{1}$
$N(\varepsilon)$	-1	-1	+1	-1	-1	+1	+1	+1	-1	+1	-1

A fast method to compute the set of square-free  $m > 1$  for which  $N(\varepsilon) = -1$  (equivalently,  $l$  is odd) is not known [18, 19, 20, 21, 22, 23]. Likewise, the set of  $m \equiv 5 \pmod{8}$  for which  $A_{l-1} + B_{l-1}\sqrt{m} = \varepsilon^3$  remains only partially understood [24, 25, 26, 27, 28, 29, 30]. Since  $\varepsilon$  can be exponentially large in  $m$ , the **regulator**  $\ln(\varepsilon)$  is often used instead [31]. Hallgren [32, 33] recently gave a polynomial-time algorithm for computing  $\ln(\varepsilon)$  that is based on a quantum Fourier transform period finding technique.

Another formula is  $\varepsilon = (x + y\sqrt{D})/2$ , where  $x, y$  are the smallest positive integer solutions of the Pell equation  $x^2 - Dy^2 = \pm 4$ . It follows immediately that  $N(\varepsilon) = -1$  if and only if  $x^2 - Dy^2 = -4$ . Let us define  $\varepsilon^+ = (z + w\sqrt{D})/2$ , where  $z, w$  are the smallest positive integer solutions of  $z^2 - Dw^2 = 4$ . Clearly  $h^+(D) \ln(\varepsilon^+) = 2h(D) \ln(\varepsilon)$  for all  $D > 0$ ; we will need  $\varepsilon^+$  later.

**0.4. Ideal Statistics over  $D$ .** The study of ideal class numbers as functions of fundamental discriminant  $D$  (equivalently, radicand  $m$ ) has occupied mathematicians for centuries. Heegner [34], Stark [35, 36, 37], Baker [38], Deuring [39] and Siegel [40, 41] solved Gauss' class number one problem:  $h(D) = 1$  for  $D = -3, -4, -7, -8, -11, -19, -43, -67, -163$  and for no other  $D < -163$ . See [42, 43, 44, 45, 46, 47, 48, 49, 50, 51] for related work in the imaginary case. With respect to the real case, Gauss conjectured that  $h(D) = 1$  for infinitely many  $D > 0$ , but a proof remains unknown.

Siegel [52, 53, 54, 55, 56] showed that

$$\ln(h(D)) \sim \ln(\sqrt{-D}) \quad \text{as } D \rightarrow -\infty,$$

$$\ln(h(D) \ln(\varepsilon)) \sim \ln(\sqrt{D}) \quad \text{as } D \rightarrow \infty$$

and the following mean value results apply [57, 58, 59, 60]:

$$\sum_{0 < -D < x} h(D) \sim \frac{c}{3\pi} x^{3/2}, \quad \sum_{0 < D < x} h(D) \ln(\varepsilon) \sim \frac{c}{6} x^{3/2}$$

as  $x \rightarrow \infty$ , where [61]

$$c = \prod_p \left( 1 - \frac{1}{p^2(p+1)} \right) = 0.8815138397\dots$$

and the infinite product is over all primes  $p$ . We may alternatively write

$$\lim_{x \rightarrow \infty} \mathbb{E} \left( \frac{h(D)}{\sqrt{-D}} \mid 0 < -D < x \right) = \frac{\pi c}{6} = 0.4615595671\dots$$

$$\lim_{x \rightarrow \infty} \mathbb{E} \left( \frac{h(D) \ln(\varepsilon)}{\sqrt{D}} \mid 0 < D < x \right) = \frac{\pi^2 c}{12} = 0.7250160726\dots$$

because  $\sum_{0 < -D < x} 1 \sim (3/\pi^2)x \sim \sum_{0 < D < x} 1$  and since partial summation contributes an additional factor of  $3/2$ .

Taniguchi [62] conjectured a second-order analog

$$\sum_{0 < -D < x} h(D)^2 \sim \frac{\pi^2 c'}{144} x^2, \quad \sum_{0 < D < x} h(D)^2 \ln(\varepsilon)^2 \sim \frac{\pi^4 c'}{576} x^2$$

as  $x \rightarrow \infty$ , where [63]

$$c' = \prod_p \left( 1 - \frac{3}{p^3} + \frac{2}{p^4} + \frac{1}{p^5} - \frac{1}{p^6} \right) = 0.6782344919\dots$$

With regard to extreme values, Granville & Soundararajan [64] suggested that perhaps

$$\max_{|D| < x} L(D) = e^\gamma (\ln \ln x + \ln \ln \ln x + c'' + o(1))$$

where  $\gamma$  is Euler's constant,

$$L(D) = \begin{cases} \frac{\pi h(D)}{\sqrt{-D}} & \text{if } D < -4, \\ \frac{2h(D) \ln(\varepsilon)}{\sqrt{D}} & \text{if } D > 4 \end{cases}$$

and

$$c'' = \int_0^1 \frac{\tanh(y)}{y} dy + \int_1^\infty \frac{\tanh(y) - 1}{y} dy = 0.8187801401\dots$$

Is it possible in any of these formulas, when  $D > 0$ , to somehow separate the class number and the regulator?

**0.5. Cohen-Lenstra Heuristics.** We merely state certain conjectures due to Cohen & Lenstra [65, 66, 67, 68, 69, 70]. Define  $\tilde{H}_m$  to be the odd part of the class group  $H_m$ , that is,  $\tilde{H}_m$  is the subgroup of all elements in  $H_m$  of odd order. Let [71, 72]

$$C = \prod_{j=2}^{\infty} \zeta(j) = 2.2948565916\dots,$$

$$\Delta = \frac{\pi^2}{6} \prod_p \left( 1 + \frac{1}{p^2(p-1)} \right) = 2.2038565964\dots$$

and, when  $q$  is prime,

$$\eta(q) = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{q^k} \right)$$

(which appeared in [73] for the special case  $q = 2$ ). For random  $m < 0$ , it is believed that

- the probability that  $\tilde{H}_m$  is cyclic is

$$\frac{\pi^2 \zeta(3)}{18 \zeta(6)} \frac{1}{C\eta(2)} = 0.9775748102\dots$$

- if  $p$  is an odd prime, the probability that  $p|h_m$  is

$$1 - \eta(p) = \begin{cases} 0.4398739220\dots & \text{if } p = 3, \\ 0.2396672041\dots & \text{if } p = 5, \\ 0.1632045929\dots & \text{if } p = 7 \end{cases}$$

and, likewise, for random  $m > 0$ ,

- the probability that  $\tilde{H}_m$  is cyclic is

$$\frac{3}{10} \frac{\Delta}{C\eta(2)} = 0.9976305717\dots$$

- if  $p$  is an odd prime, the probability that  $p|h_m$  is

$$1 - \left(1 - \frac{1}{p}\right)^{-1} \eta(p) = \begin{cases} 0.1598108831\dots & \text{if } p = 3, \\ 0.0495840051\dots & \text{if } p = 5, \\ 0.0237386917\dots & \text{if } p = 7 \end{cases}$$

- the probability that  $h_m = 1$ , given that  $m$  itself is prime, is

$$\frac{1}{2C\eta(2)} = 0.7544581722\dots$$

A proof of any of these conjectures would be a welcome breakthrough! See [74] for partial results concerning the prime  $p = 3$ .

**0.6. Form Statistics over  $d$ .** Given a nonsquare discriminant  $d$ , define  $h^+(d)$  and  $\varepsilon^+(d)$  exactly as before (with  $D$  simply replaced by  $d$ ). We had no need of such generalizations until now. See Table 6 [75].

Table 6 *Class Number  $h^+(d)$  for  $-23 \leq d \leq 32$ ; also  $\varepsilon^+(d)$  for  $5 \leq d \leq 32$*

$d$	-3	-4	-7	-8	-11	-12	-15	-16	-19	-20	-23
$h^+(d)$	1	1	1	1	1	1	2	1	1	2	3
$d$	5	8	12	13	17	20	21	24	28	29	32
$h^+(d)$	1	1	2	1	1	1	2	2	2	1	2
$\varepsilon^+(d)$	$\frac{3+\sqrt{5}}{2}$	$\frac{3+2\sqrt{2}}{1}$	$\frac{2+\sqrt{3}}{1}$	$\frac{11+3\sqrt{13}}{2}$	$\frac{33+8\sqrt{17}}{1}$	$\frac{9+4\sqrt{5}}{1}$	$\frac{5+\sqrt{21}}{2}$	$\frac{5+2\sqrt{6}}{1}$	$\frac{8+3\sqrt{7}}{1}$	$\frac{27+5\sqrt{29}}{2}$	$\frac{3+2\sqrt{2}}{1}$

Lipschitz [76], Mertens [77] and Siegel [78] proved that

$$\sum_{0 < -d < x} h^+(d) \sim \frac{\pi}{18\zeta(3)} x^{3/2}, \quad \sum_{0 < d < x} h^+(d) \ln(\varepsilon^+) \sim \frac{\pi^2}{18\zeta(3)} x^{3/2}$$

as  $x \rightarrow \infty$ , where the sums are taken over all  $d \equiv 0, 1 \pmod{4}$  that are not squares. Their efforts confirmed conjectures of Gauss [79, 80, 81, 82]:

$$\sum_{\substack{0 < -d < 4x, \\ 4|d}} h^+(d) \sim \frac{4\pi}{21\zeta(3)} x^{3/2}, \quad \sum_{\substack{0 < d < 4x, \\ 4|d}} h^+(d) \ln(\varepsilon^+) \sim \frac{4\pi^2}{21\zeta(3)} x^{3/2}.$$

When searching through the literature, it is helpful to be aware of Gauss's convention (that  $d = 4k$  or, equivalently,  $f(x, y) = ax^2 + 2bxy + cy^2$ ) versus Eisenstein's convention (no parity requirement on the middle coefficient). We have adopted the latter, as do most contemporary authors. For example,

$$\lim_{x \rightarrow \infty} \mathbb{E} \left( \frac{h^+(d)}{\sqrt{-d}} \mid 0 < -d < 4x, d = 4k \right) = \frac{\pi}{7\zeta(3)} = 0.3733591557\dots = \frac{1.1729423808\dots}{\pi}$$

in Gauss' scheme and

$$\lim_{x \rightarrow \infty} \mathbb{E} \left( \frac{h^+(d) \ln(\varepsilon^+)}{\sqrt{d}} \mid 0 < d < x \right) = \frac{\pi^2}{6\zeta(3)} = 1.3684327776\dots = 2(0.6842163888\dots)$$

in Eisenstein's scheme. A second-moment analog of the latter is due to Barban [83, 84, 85, 86, 87, 88, 89]:

$$\begin{aligned} \lim_{x \rightarrow \infty} \mathbb{E} \left( \frac{h^+(d)^2 \ln(\varepsilon^+)^2}{d} \mid 0 < d < x \right) &= \prod_p \left( 1 + \frac{3p^2 - 1}{(p^2 - 1)p(p + 1)} \right) \\ &= 2.5965362904\dots = \frac{29}{18}(1.6116432147\dots) \end{aligned}$$

In fact, the probability distributions [90, 91, 92, 93, 94, 95]

$$\begin{aligned} \lim_{x \rightarrow \infty} \mathbb{P} \left\{ \ln \left( \frac{h^+(d) \ln(\varepsilon^+)}{\sqrt{d}} \right) \leq s \mid 0 < d < x \right\}, \\ \lim_{x \rightarrow \infty} \mathbb{P} \left\{ \ln \left( \frac{\pi h^+(d)}{\sqrt{-d}} \right) \leq s \mid 0 < -d < x \right\} \end{aligned}$$

both coincide with the distribution of  $S = \sum_p X_p$ , an infinite sum of independent random variables, where

$$X_p = \begin{cases} 0 & \text{with probability } \frac{1}{p}, \\ -\ln \left( 1 - \frac{1}{p} \right) & \text{with probability } \frac{1}{2} \left( 1 - \frac{1}{p} \right), \\ -\ln \left( 1 + \frac{1}{p} \right) & \text{with probability } \frac{1}{2} \left( 1 - \frac{1}{p} \right) \end{cases}$$

for each prime number  $p$ .

We mention finally Hooley's conjecture [96]

$$\sum_{\substack{0 < d < 4x, \\ 4|d}} h^+(d) \sim \frac{25}{12\pi^2} x \ln(x)^2$$

and wonder if this (and other attempts to separate the class number and the regulator when  $d > 0$ ) someday can be verified.

**0.7. Continued Fraction Period Length.** Table 7 exhibits the period length  $l_m$  of the continued fraction expansion for  $\sqrt{m}$ , where  $m > 1$  is square-free [97].

Table 7 *Period Length as a Function of  $m$ ,  $2 \leq m \leq 31$*

$m$	2	3	5	6	7	10	11	13	14	15	17	19	21	22	23	26	29	30	31
$l_m$	1	2	1	2	4	1	2	5	4	2	1	6	6	6	4	1	5	2	8

Very little can be said about the behavior of  $l_m$ . Podsypanin [98, 99] proved that

$$l_m = O\left(\sqrt{m} \ln(\ln(m))\right)$$

as  $m \rightarrow \infty$ , assuming the truth of the extended Riemann hypothesis. Williams [100, 101] gave evidence that the big  $O$ , on the one hand, can be replaced by

$$\frac{e^\gamma}{\ln(\varphi)} = 3.7012232975\dots$$

where  $\varphi$  is the Golden mean, or even

$$\frac{12e^\gamma \ln(2)}{\pi^2} = 1.5010271229\dots$$

It seems likely, on the other hand, that the values 1.05 or even 1.08 will *not* suffice. Pen & Skubenko [102] and Golubeva [103, 104] proved the inequality [105]

$$\frac{\ln(\varepsilon)}{\ln(4\sqrt{m})} < l_m < \frac{4 \ln(\varepsilon)}{\ln(\varphi)} = 4(2.0780869212\dots) \ln(\varepsilon)$$

involving the fundamental unit  $\varepsilon$  of  $\mathbb{Q}(\sqrt{m})$ . This subject turns out to be related to what are called **Lévy constants** [106, 107, 108, 109]:

$$\beta(\xi) = \lim_{k \rightarrow \infty} \frac{\ln(B_k)}{k}$$

where  $A_k/B_k$  is the  $k^{\text{th}}$  convergent of the quadratic irrational  $\xi$ . Let  $\Sigma$  denote the set of all such  $\beta(\xi)$ . It is known that  $\Sigma \subseteq [\ln(\varphi), \infty)$  and that  $\pi^2/(12 \ln(2))$  is a limit point of  $\Sigma$ . It is also likely that  $\Sigma$  has a structure similar to the Markov spectrum [110] in the sense that a left hand portion of  $\Sigma$  probably consists only of isolated points and a right hand portion of  $\Sigma$  is much denser.

Let  $3 < p \equiv 3 \pmod{4}$  be prime and assume that  $h_p = 1$ . An astonishing formula due to Hirzebruch [111, 112, 113, 114] states that

$$h_{-p} = \frac{1}{3} \sum_{j=1}^l (-1)^{l-j} d_j$$

where  $d_1, d_2, \dots, d_l$  is the sequence of denominators in one period of the continued fraction expansion for  $\sqrt{p} - \lfloor \sqrt{p} \rfloor$ . For example,  $h_{23} = 1$  and  $h_{-23} = (-1 + 3 - 1 + 8)/3 = 3$ . Is an elementary proof of this theorem possible? What can be said if instead  $p \equiv 1 \pmod{4}$ ?

As an aside, there exist precisely twenty-one square-free integers  $m$  for which the pair  $(\mathcal{O}_m, |N|)$  is a Euclidean domain, that is, for which  $|N|$  is compatible with the division algorithm [16, 115, 116, 117, 118]. Both  $(\mathcal{O}_{14}, |N|)$  and  $(\mathcal{O}_{69}, |N|)$  fail to be Euclidean, although  $h_{14} = 1 = h_{69}$ . An alternative function  $N' : \mathcal{O}_{69} \rightarrow \mathbb{Z}$  can be constructed so that  $(\mathcal{O}_{69}, |N'|)$  is Euclidean [119, 120, 121, 122, 123]; the proof turns out to be computer-assisted. Does such a construction exist for  $\mathcal{O}_{14}$  [124, 125]?

As another aside,  $h(j^2 + 4) > 1$  for odd  $j > 17$  and  $h(4k^2 + 1) > 1$  for  $k > 13$ . The arguments  $j^2 + 4$  and  $4k^2 + 1$  are assumed to be square-free. These two inequalities, known respectively as Yokoi's conjecture and Chowla's conjecture, were proved only recently by Biró [126, 127, 128, 129, 130].

We have not discussed prime-producing polynomials [131], asymptotic  $h(d)$ -averages over subsets [132, 133], the theory of genera [1] or Dirichlet L-series, although the definition of  $L(D)$  earlier provides some foreshadowing of an upcoming essay [134].

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