

Central Binomial Coefficients

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The largest coefficient of the polynomial $(1+x)^n$ is [1]

$$A(n) = \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$

It possesses recursion

$$\left\lceil \frac{n+1}{2} \right\rceil A(n+1) = (n+1)A(n), \quad A(0) = 1$$

and asymptotics

$$A(n) \sim \sqrt{\frac{2}{\pi}} n^{-1/2} 2^n$$

as $n \rightarrow \infty$. Another interpretation of $A(n)$ is as the number of sign choices $+$ and $-$ such that

$$\begin{cases} \pm 1 \pm 1 \pm 1 \pm \cdots \pm 1 = 0 & \text{if } n \text{ is even,} \\ \underbrace{\pm 1 \pm 1 \pm 1 \pm \cdots \pm 1}_n = 1 & \text{if } n \text{ is odd.} \end{cases}$$

The latter is an especially attractive characterization of the n^{th} central binomial coefficient.

Contrast this with the n^{th} central trinomial coefficient, $B(n)$, defined to be the largest coefficient of the polynomial $(1+x+x^2)^n$. There is no simple closed-form expression for $B(n)$ [2]. It possesses recursion

$$(n+1)B(n+1) = (2n+1)B(n) + 3nB(n-1), \quad B(0) = B(1) = 1$$

and asymptotics

$$B(n) \sim \sqrt{\frac{3}{4\pi}} n^{-1/2} 3^n.$$

Here, $B(n)$ can be interpreted as the number of solutions of

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_n = 0$$

where each $\varepsilon_j \in \{-1, 0, 1\}$. Easy proofs of the asymptotics of $A(n)$ and $B(n)$ can be based on such additive representations, coupled with the Central Limit Theorem [3].

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0.1. Divisibility. Let $\omega(n, k)$ denote the number of distinct prime factors of $\binom{n}{k}$. Erdős [4, 5] proved that

$$\omega(2n, n) \sim 2 \ln(2) \frac{n}{\ln(n)}$$

as $n \rightarrow \infty$ and wondered what else could be said about the prime factors. Let

$$f(n) = \sum_{\substack{p \leq n, \\ p \mid \binom{2n}{n}}} \frac{1}{p},$$

then [6, 7]

$$c = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n) = \sum_{k=2}^{\infty} \frac{\ln(k)}{2^k} = 0.5078339228\dots,$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (f(n) - c)^2 = 0.$$

These two facts together express that $f(n) \rightarrow c$ for almost all integers n , hence $\binom{2n}{n}$ is almost always divisible by high powers of small primes. Let $g(n)$ be the smallest odd prime factor of $\binom{2n}{n}$. Whether $f(n)$ or $g(n)$ are bounded remains an open question.

Sárközy [8] and others [9, 10, 11] proved that $\binom{2n}{n}$ is not square-free for any $n > 4$. The largest n for which $\binom{2n}{n}$ is not divisible by p^2 for any odd prime p is $n = 786$.

We turn attention to $\binom{n}{k}$, the $(k+1)$ st element in the n th row of Pascal's triangle. For each $k \geq 1$, the sequence of integers n such that $\binom{n}{k}$ is square-free has asymptotic density c_k , where

$$c_1 = \frac{6}{\pi^2} = 0.6079271018\dots, \quad c_2 = \frac{3}{4} \prod_{p \geq 3} \left(1 - \frac{2}{p^2}\right) = 0.4839511484\dots$$

(the latter is related to the Feller-Tornier constant [12]). More generally, write k in base p :

$$k = a_0 + a_1 p + a_2 p^2 + \dots + a_\ell p^\ell, \quad 0 \leq a_j < p \text{ for all } 0 \leq j \leq \ell, \quad a_{\ell+1} = 0,$$

and define

$$c_{k,p} = \begin{cases} \prod_{i=0}^{\ell} \left(1 - \frac{a_i}{p}\right) \cdot \left(1 + \sum_{j=0}^{\ell} \frac{a_j(p-1-a_{j+1})}{(p-a_j)(p-a_{j+1})}\right) & \text{if } p \leq k, \\ 1 - \frac{k}{p^2} & \text{if } p > k. \end{cases}$$

Then c_k is equal to $\prod_p c_{k,p}$, where the product is taken over all primes p . We have $c_3 = 0.251\dots$, $c_4 = 0.360\dots$, $c_5 = 0.191\dots$, $c_6 = 0.189\dots$, $c_7 = 0.062\dots$ and

$$0 < c_k = \exp \left[-(\alpha + o(1))\sqrt{k}/\ln(k) \right]$$

as $k \rightarrow \infty$, where

$$\begin{aligned} \alpha &= \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \int_0^{\infty} \{x\}^j x^{-3/2} dx \\ &= \sum_{j=1}^{\infty} \binom{2j}{j} \zeta(j+1/2) \frac{1}{2^{2j-1}} \left(1 - j \sum_{i>j} \frac{1}{i^2} \right) \\ &= 1.825108\dots \end{aligned}$$

Integrals involving $\{x\} = x - [x]$ as such also appear in [13, 14]. It follows that there are $\sim \tau N$ square-free binomial coefficients $\binom{n}{k}$ with $0 \leq k < n \leq N$, where

$$\tau = 2 \sum_{k=0}^{\infty} c_k = 2(5.3275\dots) = 10.655\dots$$

In words, each row of Pascal's triangle possesses approximately $10\frac{2}{3}$ square-free entries (on average).

0.2. Relevant Sums. Let φ denote the Golden mean $(1 + \sqrt{5})/2$. We have [15, 16, 17, 18, 19]

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} &= \frac{1}{3} + \frac{2\sqrt{3}\pi}{27}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}} &= \frac{1}{5} + \frac{4\sqrt{5}\ln(\varphi)}{25}, \\ \sum_{n=1}^{\infty} \frac{n}{\binom{2n}{n}} &= \frac{2}{3} + \frac{2\sqrt{3}\pi}{27}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{\binom{2n}{n}} &= \frac{6}{25} + \frac{4\sqrt{5}\ln(\varphi)}{125}, \\ \sum_{n=1}^{\infty} \frac{n^2}{\binom{2n}{n}} &= \frac{4}{3} + \frac{10\sqrt{3}\pi}{81}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{\binom{2n}{n}} &= \frac{4}{25} - \frac{4\sqrt{5}\ln(\varphi)}{125}, \\ \sum_{n=1}^{\infty} \frac{n^3}{\binom{2n}{n}} &= \frac{10}{3} + \frac{74\sqrt{3}\pi}{243}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^3}{\binom{2n}{n}} &= -\frac{2}{125} - \frac{28\sqrt{5}\ln(\varphi)}{625} \end{aligned}$$

and, more generally [20],

$$\sum_{n=1}^{\infty} \frac{n^k}{\binom{2n}{n}} = p_k + q_k \sqrt{3}\pi, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^k}{\binom{2n}{n}} = r_k + s_k \sqrt{5}\ln(\varphi)$$

for appropriate rationals p_k, q_k, r_k, s_k . Let L_D denote the Dirichlet L-series with character (D/\cdot) and Li_k denote the k^{th} polylogarithm function. The following are more difficult [15, 16, 17, 18, 19]:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}n} &= \frac{\sqrt{3}\pi}{9}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}n} &= \frac{2\sqrt{5}\ln(\varphi)}{5}, \\ \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}n^2} &= \frac{\pi^2}{18}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}n^2} &= 2\ln(\varphi)^2, \\ \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}n^3} &= \frac{\sqrt{3}\pi}{2}L_{-3}(2) - \frac{4\zeta(3)}{3}, & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}n^3} &= \frac{2\zeta(3)}{5}, \\ & \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}n^4} &= \frac{17\pi^4}{3240}, \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}n^4} &= 8\text{Li}_4\left(\frac{1}{\varphi}\right) + 8\ln(\varphi)\text{Li}_3\left(\frac{1}{\varphi}\right) - \frac{1}{2}\text{Li}_4\left(\frac{1}{\varphi^2}\right) \\ & \quad + \frac{7\pi^2\ln(\varphi)^2}{15} - \frac{13\ln(\varphi)^4}{6} - \frac{4\zeta(3)\ln(\varphi)}{5} - \frac{7\pi^4}{90}, \\ \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}n^5} &= \frac{9\sqrt{3}\pi}{8}L_{-3}(4) + \frac{\pi^2\zeta(3)}{9} - \frac{19\zeta(5)}{3}, \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}n^5} &= \frac{5}{2}\text{Li}_5\left(\frac{1}{\varphi^2}\right) + 5\ln(\varphi)\text{Li}_4\left(\frac{1}{\varphi^2}\right) \\ & \quad + 4\zeta(3)\ln(\varphi)^2 - \frac{4\pi^2\ln(\varphi)^3}{9} + \frac{4\ln(\varphi)^5}{3} - 2\zeta(5). \end{aligned}$$

We wonder whether the last two alternating series possess expressions involving L-series values rather than polylogarithmic values. Let $G = L_{-4}(2)$ denote Catalan's constant. Other series include

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)} &= \frac{2\sqrt{3}\pi}{9}, & \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)} &= \frac{4\sqrt{5}\ln(\varphi)}{5}, \\ \sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} &= \frac{8G}{3} - \frac{\pi\ln(2+\sqrt{3})}{3}, & \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^2} &= \frac{\pi^2}{6} - 3\ln(\varphi)^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2^n}{\binom{2n}{n}(2n+1)} &= \frac{\pi}{2}, & \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)} &= \frac{2}{\sqrt{3}} \ln \left(\frac{1+\sqrt{3}}{\sqrt{2}} \right), \\ \sum_{n=0}^{\infty} \frac{2^n}{\binom{2n}{n}(2n+1)^2} &= 2L_{-8}(2) - \frac{\sqrt{2}\pi}{4} \ln(1+\sqrt{2}), \\ \sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{2n}{n}(2n+1)^2} &= 2G, & \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{\binom{2n}{n}(2n+1)^2} &= \frac{\pi^2}{8} - \frac{1}{2} \ln(1+\sqrt{2})^2, \end{aligned}$$

but similar expressions for

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^3}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^3}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)^2}$$

remain open (as far as is known).

Batir [21, 22] recently proved that

$$\sum_{n=1}^{\infty} \frac{2^{4n}}{\binom{2n}{n}^2 n^3} = 8\pi G - 14\zeta(3), \quad \sum_{n=0}^{\infty} \frac{2^{4n+2}}{\binom{2n}{n}^2 (2n+1)^3} = 14\zeta(3) - 4\pi G$$

and also derived a complicated formula for $\sum_{n=1}^{\infty} 1/\binom{3n}{n}$. We will barely mention cases for which $\binom{2n}{n}$ is in the numerator, for example [15, 17, 23],

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}(2n+1)} &= \frac{\pi}{2}, \\ \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}(2n+1)^2} &= \frac{\pi \ln(2)}{2}, & \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^2} &= \frac{\sqrt{2}}{8} (\pi \ln(2) + 4G), \\ \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^2} &= \frac{3\sqrt{3}}{4} L_{-3}(2), \\ \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^3} &= \frac{7\pi^3}{216}, & \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^4} &= \frac{\pi\zeta(3)}{12} + \frac{27\sqrt{3}}{64} L_{-3}(4), \\ \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{1}{2^{4n}(2n+1)} &= \frac{4G}{\pi}. \end{aligned}$$

Similar expressions for

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^3}, \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^4}, \quad \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{(-1)^n}{2^{4n}(2n+1)}$$

remain open (as far as is known). Techniques in [24] might be helpful in evaluating sums as these.

0.3. Addendum. Let χ_k denote the k^{th} Legendre chi function:

$$\chi_k(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)^k} = \frac{1}{2} (\text{Li}_k(x) - \text{Li}_k(-x)) = i \text{Ti}_k(-ix).$$

Gosper [25] evaluated one of the preceding open sums:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n} (2n+1)^2} = \frac{\pi^2 - 2 \operatorname{arccosh}(7) \operatorname{arccoth}(\sqrt{2+\sqrt{3}}) - 8\chi_2(\sqrt{2-\sqrt{3}})}{2\sqrt{2}}$$

which is apparently new. The other sums all possess multiple Li_k terms, for example,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n} (2n+1)^3} &= \frac{19}{3} \ln(\varphi)^3 - 2 \ln(2) \ln(\varphi)^2 - 4 \ln(\varphi) \text{Li}_2\left(\frac{\varphi}{2}\right) \\ &\quad - 4 \ln(2)^2 \ln(\varphi) + \frac{7\pi^2 \ln(\varphi)}{10} - \text{Li}_3\left(\frac{1}{2} - \frac{\varphi}{4}\right) \\ &\quad + 4 \text{Li}_3\left(1 - \frac{\varphi}{2}\right) - 3 \text{Li}_3\left(\frac{1}{\varphi}\right) + 4 \text{Li}_3\left(\frac{1}{2\varphi}\right) + \frac{3\zeta(3)}{10} \end{aligned}$$

and further simplification seems to be impossible.

Deninger's conjecture [26]

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{1}{2^{8n} (2n+1)} = \frac{15}{\pi^2} L_{15A}(2)$$

was recently proved by Rogers & Zudilin [27], where L_{15A} is the L-series for the elliptic curve isogeny class 15A. See [28] for a sampling of other conjectures.

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