

## Variants of Brownian Motion

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We defined standard Brownian motion  $\{W_t : t \geq 0\}$  in [1]. An alternative characterization of the Wiener process involves the limit of random walks. Let  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  be a sequence of independent identically distributed random variables, each possessing mean 0 and variance 1. Let

$$S_0 = 0, \quad S_1 = \varepsilon_1, \quad S_2 = \varepsilon_1 + \varepsilon_2, \quad \dots, \quad S_n = \sum_{k=1}^n \varepsilon_k.$$

Then the random walk  $\{S_k\}_{k=1}^n$  approaches Brownian motion on the unit interval in the sense that

$$\frac{S_{[nt]}}{\sqrt{n}} \rightarrow W_t, \quad 0 \leq t \leq 1$$

as  $n \rightarrow \infty$ , via the functional central limit theorem of Donsker [2, 3]. We are interested in the  $L_p$ -norm of Brownian motion

$$\|W\|_p = \begin{cases} \left( \int_0^1 |W_t|^p dt \right)^{1/p} & \text{if } 0 < p < \infty, \\ \max_{0 \leq t \leq 1} |W_t| & \text{if } p = \infty \end{cases}$$

for a number of reasons [4, 5]. Note that  $\|W\|_p$  is itself a random variable. A distributional statement about  $\|W\|_p$  hence translates into an asymptotic distributional statement about the  $l_p$ -norm of the random walk:

$$\mathbb{P} \left( \|W\|_p \leq x \right) = \begin{cases} \lim_{n \rightarrow \infty} \mathbb{P} \left( \left( \sum_{k=1}^n |S_k|^p \right)^{1/p} \leq n^{\frac{1}{2} + \frac{1}{p}} x \right) & \text{if } 0 < p < \infty, \\ \lim_{n \rightarrow \infty} \mathbb{P} \left( \max \{ |S_1|, |S_2|, \dots, |S_n| \} \leq n^{1/2} x \right) & \text{if } p = \infty. \end{cases}$$

In the following sections, we will discuss the cases  $p = \infty, 1$  and  $2$  for several variants of Brownian motion. Corresponding problems for all other values of  $p > 0$  remain unsolved.

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Some preliminary definitions include

$$\delta_m = \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi}m!} = \begin{cases} \frac{1 \cdot 3 \cdot 5 \cdots (2m - 1)}{2 \cdot 4 \cdot 6 \cdots (2m)} & \text{if } m \geq 1, \\ 1 & \text{if } m = 0, \end{cases}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt = 1 - \operatorname{erfc}(x),$$

$$\operatorname{Ai}(x) = \begin{cases} \frac{1}{3}(-x)^{1/2} \left[ J_{-\frac{1}{3}}\left(\frac{2}{3}(-x)^{3/2}\right) + J_{\frac{1}{3}}\left(\frac{2}{3}(-x)^{3/2}\right) \right] & \text{if } x < 0, \\ \frac{1}{3}x^{1/2} \left[ I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) \right] & \text{if } x \geq 0, \end{cases}$$

$$K_{\frac{1}{4}}(x) = \frac{\pi}{\sqrt{2}} \left[ I_{-\frac{1}{4}}(x) - I_{\frac{1}{4}}(x) \right]$$

where  $J_\nu(x)$  and  $I_\nu(x)$  are the well-known Bessel functions. Also, for  $x > 0$  and  $0 < a < b$ , let

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tx} t^{a-1} (1+t)^{b-a-1} dt.$$

This is called the confluent hypergeometric function of the second kind (in contrast to [1]). Finally, define the Riemann xi function

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(\frac{1}{2}z)\zeta(z), \quad \operatorname{Re}(z) > 1,$$

which serves as a tantalizing link between Brownian motion and number theory [6]. This can be analytically continued to an entire function via functional equation  $\xi(z) = \xi(1-z)$ .

**0.1. Bridge.** A Brownian bridge  $\{X_t : 0 \leq t \leq 1\}$  has the same distribution as  $\{W_t : 0 \leq t \leq 1\}$ , conditioned on  $W_1 = 0$ . The maximum of  $|X_t|$  turns out to be closely allied with the Kolmogorov-Smirnov goodness-of-fit test [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]:

$$\mathbb{P}(\|X\|_\infty \leq x) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2x^2} = \frac{\sqrt{2\pi}}{x} \sum_{k=0}^{\infty} e^{-\pi^2(2k+1)^2/(8x^2)}$$

(and the right-hand equality follows via Poisson summation). This distribution has moments

$$\mathbb{E}(\|X\|_\infty) = \sqrt{\frac{\pi}{2}} \ln(2), \quad \mathbb{E}(\|X\|_\infty^2) = \frac{\pi^2}{12}$$

and median 0.8275735551.... It also satisfies [17, 18]

$$\mathbb{E}(\|X\|_\infty^z) = 2 \frac{1 - 2^{1-z}}{z - 1} \left(\frac{\pi}{2}\right)^{z/2} \xi(z)$$

for all complex  $z$ .

Takács [19, 20], building on Cifarelli [21], Shepp [22], Rice [23] and Johnson & Killeen [24], computed that

$$\mathbb{P}(\|X\|_1 \leq x) = \frac{\sqrt{\pi}}{18^{1/6}x} \sum_{j=1}^{\infty} e^{-u_j} u_j^{-1/3} \text{Ai}\left((3u_j/2)^{2/3}\right)$$

for  $x > 0$ , where  $u_j = (a'_j)^3/(27x^2)$  and  $0 < a'_1 < a'_2 < \dots$  are the zeroes [25] of  $\text{Ai}'(-x)$ . This distribution has moments

$$\mathbb{E}(\|X\|_1) = \frac{1}{4}\sqrt{\frac{\pi}{2}}, \quad \mathbb{E}(\|X\|_1^2) = \frac{7}{60}$$

and median 0.2817802658....

Anderson & Darling [26, 27, 28, 29], building on Smirnov [30], obtained that

$$\mathbb{P}(\|X\|_2^2 \leq x) = \frac{1}{\pi\sqrt{x}} \sum_{j=0}^{\infty} \sqrt{4j+1} e^{-(4j+1)^2/(16x)} \delta_j K_{1/4}\left((4j+1)^2/(16x)\right),$$

which has moments

$$\mathbb{E}(\|X\|_2^2) = \frac{1}{6}, \quad \mathbb{E}(\|X\|_2^4) = \frac{1}{20}$$

and median 0.1188795509.... The  $L_2$ -norm, squared, of  $X_t$  turns out to be closely allied with the Cramér-von Mises goodness-of-fit test [31, 32, 33].

**0.2. Excursion.** A Brownian excursion  $\{Y_t : 0 \leq t \leq 1\}$  has the same distribution as  $\{W_t : 0 \leq t \leq 1\}$ , conditioned on  $W_t > 0$  for all  $0 < t < 1$  and  $W_1 = 0$ .

Chung [34, 35], Kennedy [36] and Durrett & Iglehart [37, 38] showed that

$$\mathbb{P}(\|Y\|_\infty \leq x) = \sum_{k=-\infty}^{\infty} (1 - 4k^2x^2)e^{-2k^2x^2} = \frac{\sqrt{2}\pi^{5/2}}{x^3} \sum_{k=1}^{\infty} k^2 e^{-\pi^2k^2/(2x^2)},$$

which has moments

$$\mathbb{E}(\|Y\|_\infty) = \sqrt{\frac{\pi}{2}}, \quad \mathbb{E}(\|Y\|_\infty^2) = \frac{\pi^2}{6},$$

median 1.2234880197..., and also satisfies [17, 18]

$$\mathbb{E}(\|Y\|_\infty^z) = 2 \left(\frac{\pi}{2}\right)^{z/2} \xi(z)$$

for all complex  $z$ .

Takács [19, 39], building on Gettoor & Sharpe [40], Darling [41], Louchard [42, 43] and Groenboom [44], obtained that

$$\mathbb{P}(\|Y\|_1 \leq x) = \frac{\sqrt{6}}{x} \sum_{j=1}^{\infty} e^{-v_j} v_j^{2/3} U\left(\frac{1}{6}, \frac{4}{3}, v_j\right)$$

for  $x > 0$ , where  $v_j = 2a_j^3/(27x^2)$  and  $0 < a_1 < a_2 < \dots$  are the zeroes [25] of  $\text{Ai}(-x)$ . This distribution has moments

$$\mathbb{E}(\|Y\|_1) = \sqrt{\frac{\pi}{8}}, \quad \mathbb{E}(\|Y\|_1^2) = \frac{5}{12}$$

and median 0.6070363869....

The  $L_2$  case seems to be open for Brownian excursion.

**0.3. Meander.** A Brownian meander  $\{Z_t : 0 \leq t \leq 1\}$  has the same distribution as  $\{W_t : 0 \leq t \leq 1\}$ , conditioned on  $W_t > 0$  for all  $0 < t < 1$ . Note that  $Z_1$  need not be zero.

Durrett & Iglehart [37, 38] computed that

$$\mathbb{P}(\|Z\|_\infty \leq x) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-k^2 x^2/2} = \frac{2^{3/2} \sqrt{\pi}}{x} \sum_{k=0}^{\infty} e^{-\pi^2(2k+1)^2/(2x^2)}.$$

Observe that the distribution of  $\|Z\|_\infty$  is the same as the distribution of  $2\|X\|_\infty$ . Hence it has moments

$$\mathbb{E}(\|Z\|_\infty) = \sqrt{2\pi} \ln(2), \quad \mathbb{E}(\|Z\|_\infty^2) = \frac{\pi^2}{3}$$

and median 1.6551471103...

Takács [45] proved that

$$\mathbb{P}(\|Z\|_1 \leq x) = \frac{\sqrt{\pi}}{18^{1/6} x} \sum_{j=1}^{\infty} b_j e^{-\tilde{v}_j} \tilde{v}_j^{-1/3} \text{Ai}\left((3\tilde{v}_j/2)^{2/3}\right)$$

for  $x > 0$ , where  $\tilde{v}_j = v_j/2$  and  $v_j, a_j$  are as before, and where

$$b_j = \frac{a_j}{3 \text{Ai}'(-a_j)} \left(1 + 3 \int_0^{a_j} \text{Ai}(-s) ds\right).$$

This distribution has moments

$$\mathbb{E}(\|Z\|_1) = \frac{3}{4}\sqrt{\frac{\pi}{2}}, \quad \mathbb{E}(\|Z\|_1^2) = \frac{59}{60}$$

and median 0.8900420723....

The  $L_2$  case seems to be open for Brownian meander.

**0.4. Motion.** We return to Brownian motion. Erdős & Kac [46, 47, 48, 49] computed that

$$\begin{aligned} \mathbb{P}(\|W\|_\infty \leq x) &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-\pi^2(2k+1)^2/(8x^2)} \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^k \left[ \operatorname{erf}\left(\frac{(2k+1)x}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{(2k-1)x}{\sqrt{2}}\right) \right] \\ &= \sum_{k=-\infty}^{\infty} \left[ \operatorname{erf}\left(\frac{(4k+1)x}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{(4k-1)x}{\sqrt{2}}\right) \right], \end{aligned}$$

which has moments

$$\mathbb{E}(\|W\|_\infty) = \sqrt{\frac{\pi}{2}}, \quad \mathbb{E}(\|W\|_\infty^2) = 2G$$

and median 1.1489732581.... This is a remarkable appearance of Catalan's constant  $G$ !

Takács [50, 51], building on Kac [52] and Schwinger [53], found that

$$\mathbb{P}(\|W\|_1 \leq y) = \sqrt{\frac{3}{\pi}} \int_0^y \frac{1}{x} \sum_{j=1}^{\infty} c_j e^{-\tilde{u}_j} \tilde{u}_j^{2/3} U\left(\frac{1}{6}, \frac{4}{3}, \tilde{u}_j\right) dx$$

for  $y > 0$ , where  $\tilde{u}_j = 2u_j$  and  $u_j, a'_j$  are as before, and where

$$c_j = \frac{1}{3a'_j \operatorname{Ai}(-a'_j)} \left( 1 + 3 \int_0^{a'_j} \operatorname{Ai}(-s) ds \right).$$

This distribution has moments

$$\mathbb{E}(\|W\|_1) = \frac{4}{3} \frac{1}{\sqrt{2\pi}}, \quad \mathbb{E}(\|W\|_1^2) = \frac{3}{8}$$

and median 0.4510953819.... We wonder whether the integral for  $\mathbb{P}(\|W\|_1 \leq y)$  can be termwise integrated.

Cameron & Martin [46, 54, 55, 56, 57] proved that

$$\mathbb{P} \left( \|W\|_2^2 \leq x \right) = \sqrt{2} \sum_{j=0}^{\infty} (-1)^j \delta_j \operatorname{erfc} \left( \frac{4j+1}{2\sqrt{2x}} \right)$$

which has moments

$$\mathbb{E} \left( \|W\|_2^2 \right) = \frac{1}{2}, \quad \mathbb{E} \left( \|W\|_2^4 \right) = \frac{7}{12},$$

median 0.2904760595... and Laplace transform

$$\mathbb{E} \left( \exp(-\lambda \|W\|_2^2) \right) = \sqrt{\sec(\sqrt{-2\lambda})}.$$

We close with two unanswered questions. Define the positive part of  $W_t$  to be  $W_t^+ = \max\{W_t, 0\}$ . Perman & Wellner [58, 59] studied the 1-norm of  $W_t^+$  and found the following double Laplace transform:

$$\int_0^{\infty} e^{-\mu\lambda} \mathbb{E} \left\{ \exp \left( -\sqrt{2}\lambda^{3/2} \|W^+\|_1 \right) \right\} d\lambda = \frac{\mu^{-1/2} \operatorname{Ai}(\mu) + \frac{1}{3} - \int_0^{\mu} \operatorname{Ai}(s) ds}{\sqrt{\mu} \operatorname{Ai}(\mu) - \operatorname{Ai}'(\mu)}$$

as well as moments:

$$\mathbb{E} \left( \|W^+\|_1 \right) = \frac{2}{3} \frac{1}{\sqrt{2\pi}}, \quad \mathbb{E} \left( \|W^+\|_1^2 \right) = \frac{17}{96}.$$

Does an explicit formula for  $\mathbb{P} \left( \|W^+\|_1 \leq x \right)$  exist? What can be said for other values of  $p > 0$ ?

#### REFERENCES

- [1] S. R. Finch, Ornstein-Uhlenbeck process, unpublished note (2004).
- [2] M. D. Donsker, An invariance principle for certain probability limit theorems, *Memoirs Amer. Math. Soc.* v. 6 (1951) n. 4, 1–12; MR0040613 (12,723a).
- [3] P. Billingsley, *Convergence of Probability Measures*, Wiley, 1968, pp. 68–80; MR0233396 (38 #1718).
- [4] S. R. Finch, Discrepancy and uniformity, unpublished note (2004).
- [5] S. R. Finch, Shapes of binary trees, unpublished note (2004).
- [6] P. Biane, J. Pitman and M. Yor, Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions, *Bull. Amer. Math. Soc.* 38 (2001) 435–465; MR1848256 (2003b:11083).

- [7] A. N. Kolmogorov, Sulla determinazione empirica di una legge di distribuzione, *Giornale dell'Istituto Italiano degli Attuari* 4 (1933) 83–91; Engl. transl. in *Selected Works, II. Probability Theory and Mathematical Statistics*, ed. A. N. Shiryaev, Kluwer, 1992, pp. 139–146.
- [8] A. N. Kolmogorov, Confidence limits for an unknown distribution function, *Annals of Math. Statist.* 12 (1941) 461–463; MR0006684 (4,25f).
- [9] N. V. Smirnov, On the deviation of the empirical distribution function (in Russian), *Mat. Sbornik* 6 (1939) 3–26; MR0001483 (1,246b).
- [10] N. V. Smirnov, On the estimation of the discrepancy between empirical curves of distribution for two independent samples (in Russian), *Vestnik Moskov. Univ.* 2 (1939) 3–16; MR0002062 (1,345f).
- [11] N. V. Smirnov, Table for estimating the goodness of fit of empirical distributions, *Annals of Math. Statist.* 19 (1948) 279–281; MR0025109 (9,599j).
- [12] W. Feller, On the Kolmogorov-Smirnov limit theorems for empirical distributions, *Annals of Math. Statist.* 19 (1948) 177–189; MR0025108 (9,599i).
- [13] J. L. Doob, Heuristic approach to the Kolmogorov-Smirnov theorems, *Annals of Math. Statist.* 20 (1949) 393–403; MR0030732 (11,43a).
- [14] M. D. Donsker, Justification and extension of Doob's heuristic approach to the Komogorov-Smirnov theorems, *Annals of Math. Statist.* 23 (1952) 277–281; MR0047288 (13,853n).
- [15] J. D. Gibbons, *Nonparametric Statistical Inference*, McGraw-Hill, 1971, pp. 75–85; MR0286223 (44 #3437).
- [16] G. Marsaglia, W. W. Tsang and J. Wang, Evaluating Kolmogorov's distribution, *J. Statist. Software* 8 (2003) i. 18; available online at <http://www.jstatsoft.org/index.php?vol=8>.
- [17] P. Biane and M. Yor, Valeurs principales associées aux temps locaux browniens, *Bull. Sci. Math.* 111 (1987) 23–101; MR0886959 (88g:60188).
- [18] J. Pitman and M. Yor, The law of the maximum of a Bessel bridge, *Elec. J. Probab.* 4 (1999) n. 15; MR1701890 (2000j:60101).
- [19] L. Takács, Random walk processes and their applications in order statistics, *Annals of Appl. Probab.* 2 (1992) 435–459; MR1161061 (93h:60114).

- [20] L. Tolmatz, Asymptotics of the distribution of the integral of the absolute value of the Brownian bridge for large arguments, *Annals of Probab.* 28 (2000) 132–139; available online at <http://www.tolmatz.net/>; MR1756000 (2002a:60068).
- [21] D. M. Cifarelli, Contributi intorno ad un test per l'omogeneità tra due campioni, *Giornale degli Economisti e Annali di Economia* 34 (1975) 233–249; MR0433704 (55 #6676).
- [22] L. A. Shepp, On the integral of the absolute value of the pinned Wiener process, *Annals of Probab.* 10 (1982) 234–239; acknowledgment of priority, 19 (1991) 1397; MR0637389 (82m:60100) and MR1112423 (92e:60159).
- [23] S. O. Rice, The integral of the absolute value of the pinned Wiener process—calculation of its probability density by numerical integration, *Annals of Probab.* 10 (1982) 240–243; MR0637390 (82m:60099).
- [24] B. McK. Johnson and T. Killeen, An explicit formula for the C.D.F. of the  $L_1$  norm of the Brownian bridge, *Annals of Probab.* 11 (1983) 807–808; MR0704570 (85b:60038).
- [25] S. R. Finch, Airy function zeroes, unpublished note (2004).
- [26] T. W. Anderson and D. A. Darling, Asymptotic theory of certain “goodness of fit” criteria based on stochastic processes, *Annals of Math. Statist.* 23 (1952) 193–212; MR0050238 (14,298h).
- [27] L. Tolmatz, On the distribution of the square integral of the Brownian bridge, *Annals of Probab.* 30 (2002) 253–269; addenda 31 (2003) 530–532; available online at <http://www.tolmatz.net/>; MR1894107 (2003g:60064).
- [28] R. Ghomrasni, On distributions associated with the generalized Lévy’s stochastic area formula, *Studia Sci. Math. Hungar.* 41 (2004) 93–100; available online at <http://www.maphysto.dk/publications/MPS-RR/2003/4.pdf>.
- [29] G. Marsaglia and J. Marsaglia, Evaluating the Anderson-Darling distribution, *J. Statist. Software* 9 (2004) i. 2; available online at <http://www.jstatsoft.org/index.php?vol=9>.
- [30] N. V. Smirnov, On the distribution of the  $\omega^2$ -criterion of von Mises (in Russian), *Mat. Sbornik* 2 (1937) 973–993; French transl. abbrev. in *C. R. Séances Acad. Sci.* 202 (1936) 449–452.
- [31] M. Knott, The distribution of the Cramér-von Mises statistic for small sample sizes, *J. Royal Statist. Soc. Ser B* 36 (1974) 430–438; MR0365883 (51 #2135).

- [32] G. R. Shorack and J. A. Wellner, *Empirical Processes with Applications to Statistics*, Wiley, 1986, pp. 142–150; MR0838963 (88e:60002).
- [33] R. B. D’Agostino and M. A. Stephens (eds.), *Goodness-of-Fit Techniques*, 1986, pp. 111–122; MR0874534 (88c:62075).
- [34] K. L. Chung, Maxima in Brownian excursions, *Bull. Amer. Math. Soc.* 81 (1975) 742–745; MR0373035 (51 #9237).
- [35] K. L. Chung, Excursions in Brownian motion, *Ark. Mat.* 14 (1976) 155–177; MR0467948 (57 #7791).
- [36] D. P. Kennedy, The distribution of the maximum Brownian excursion, *J. Appl. Probab.* 13 (1976) 371–376; MR0402955 (53 #6769).
- [37] R. T. Durrett, D. L. Iglehart and D. R. Miller, Weak convergence to Brownian meander and Brownian excursion, *Annals of Probab.* 5 (1977) 117–129; MR0436353 (55 #9300).
- [38] R. T. Durrett and D. L. Iglehart, Functionals of Brownian meander and Brownian excursion, *Annals of Probab.* 5 (1977) 130–135; MR0436354 (55 #9301).
- [39] L. Takács, A Bernoulli excursion and its various applications, *Adv. Appl. Probab.* 23 (1991) 557–585; MR1122875 (92m:60057).
- [40] R. K. Gettoor and M. J. Sharpe, Excursions of Brownian motion and Bessel processes, *Z. Wahrsch. Verw. Gebiete* 47 (1979) 83–106; MR0521534 (80b:60104).
- [41] D. A. Darling, On the supremum of a certain Gaussian process, *Annals of Probab.* 11 (1983) 803–806; MR0704564 (84f:60058).
- [42] G. Louchard, Kac’s formula, Levy’s local time and Brownian excursion, *J. Appl. Probab.* 21 (1984) 479–499; MR0752014 (86f:60100).
- [43] G. Louchard, The Brownian excursion area: a numerical analysis, *Comput. Math. Appl.* 10 (1984) 413–417; erratum, *A12* (1986) 375; MR0783514 (87b:60123a) and MR0837285 (87b:60123b).
- [44] P. Groeneboom, Brownian motion with a parabolic drift and Airy functions, *Probab. Theory Related Fields* 81 (1989) 79–109; MR0981568 (90c:60052).
- [45] L. Takács, Limit distributions for the Bernoulli meander, *J. Appl. Probab.* 32 (1995) 375–395; MR1334893 (96c:60087).

- [46] P. Erdős and M. Kac, On certain limit theorems of the theory of probability, *Bull. Amer. Math. Soc.* 52 (1946) 292–302; MR0015705 (7,459b).
- [47] K. L. Chung, On the maximum partial sums of sequences of independent random variables, *Trans. Amer. Math. Soc.* 64 (1948) 205–233; MR0026274 (10,132b).
- [48] M. Kac, On deviations between theoretical and empirical distributions, *Proc. Nat. Acad. Sci. USA* 35 (1949) 252–257; MR0029490 (10,614b).
- [49] A. Rényi, On the distribution function  $L(z)$  (in Hungarian), *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 2 (1957) 43–50; Engl. transl. in *Selected Transl. Math. Statist. and Probab.*, v. 4, Inst. Math. Stat. and Amer. Math. Soc., 1963, pp. 219–224; MR0099710 (20 #6148).
- [50] L. Takács, On the distribution of the integral of the absolute value of the Brownian motion, *Annals of Appl. Probab.* 3 (1993) 186–197; MR1202522 (93i:60154).
- [51] L. Tolmatz, Asymptotics of the distribution of the integral of the absolute value of the Brownian bridge for large arguments, *Annals of Probab.* 28 (2000) 132–139; available online at <http://www.tolmatz.net/>; MR1756000 (2002a:60068).
- [52] M. Kac, On the average of a certain Wiener functional and a related limit theorem in calculus of probability, *Trans. Amer. Math. Soc.* 59 (1946) 401–414; MR0016570 (8,37b).
- [53] M. Kac, *Probability, Number Theory, and Statistical Physics. Selected Papers*, ed. K. Baclawski and M. D. Donsker, MIT Press, 1979, pp. ix–xxiii; MR0521078 (80i:01016).
- [54] R. H. Cameron and W. T. Martin, The Wiener measure of Hilbert neighborhoods in the space of real continuous functions, *J. Math. Phys. Mass. Inst. Tech.* 23 (1944) 195–209; MR0011174 (6,132a).
- [55] R. Fortet, Quelques travaux récents sur le mouvement brownien, *Annales de l'Institut Henri Poincaré* 11 (1949) 175–226; MR0035411 (11,731e).
- [56] P. Lévy, Wiener's random function, and other Laplacian random functions, *Proc. Second Berkeley Symp. Math. Stat. Probab.*, ed. J. Neyman, Univ. of Calif. Press, 1951, pp. 171–187; MR0044774 (13,476b).
- [57] A. N. Borodin and P. Salminen, *Handbook of Brownian motion—Facts and Formulae*, 2<sup>nd</sup> ed., Birkhäuser Verlag, 2002, pp. 168–169, 333, 343–344; MR1912205 (2003g:60001).

- [58] M. Perman and J. A. Wellner, On the distribution of Brownian areas, *Annals of Appl. Probab.* 6 (1996) 1091–1111; MR1422979 (97m:60119).
- [59] L. Tolmatz, Asymptotics of the distribution of the integral of the positive part of the Brownian bridge for large arguments, *J. Math. Anal. Appl.* 304 (2005) 668–682; available online at <http://www.tolmatz.net/>; MR2126559 (2006b:60179).