

## Prandtl-Blasius Flow

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The boundary value problem

$$y'''(x) + y''(x)y(x) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad \lim_{x \rightarrow \infty} y'(x) = 1$$

arises in the study of two-dimensional incompressible viscous flow past a thin semi-infinite flat plate [1, 2, 3, 4]. Such an equation is similar to the Thomas-Fermi equation [5], but is even more difficult to solve (because it is of higher order).

A well-known series for  $y(x)$  is

$$y(x) = \sum_{k=0}^{\infty} (-1)^k \frac{p_k \xi^{k+1}}{(3k+2)!} x^{3k+2}, \quad x \approx 0$$

where  $p_0 = 1$  and [6]

$$p_k = \sum_{j=0}^{k-1} \binom{3k-1}{3j} p_j p_{k-j-1}, \quad k \geq 1.$$

Hence it is important to compute

$$\xi = \lim_{x \rightarrow 0^+} \frac{y(x) - 0}{x^2/2} = y''(0)$$

as accurately as feasible. Blasius' series has only a finite radius of convergence [7, 8, 9, 10, 11]:

$$\rho = \lim_{k \rightarrow \infty} \left( \frac{(3k)(3k+1)(3k+2)p_{k-1}}{p_k \xi} \right)^{1/3} = 4.0234644935\dots$$

(in fact, the associated singularities are at  $x = -\rho$  and  $\rho \exp(\pm i\pi/3)$ ). Unlike the Thomas-Fermi equation, an efficient transformation for the Blasius equation is not yet known that permits high-precision estimates of  $\xi$ . A Runge-Kutta numerical ODE solver gives  $\xi = 0.4695999883\dots$ , as well as [2, 3, 10, 11]

$$\eta = \lim_{x \rightarrow \infty} (x - y(x)) = 1.2167806216\dots$$

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The fluid dynamics literature is somewhat bewildering because of (small) variations in the presentation of Blasius' equation. Let us generalize our discussion to clear up any confusion. Consider

$$z'''(x) + a z''(x)z(x) = 0, \quad z(0) = 0, \quad z'(0) = 0, \quad \lim_{x \rightarrow \infty} z'(x) = b$$

where  $a > 0, b > 0$ . Let  $c = z''(0)$ ; it can be easily shown that  $c = a^{1/2}b^{3/2}\xi$  and thus

$$c(a = 1/2, b = 1) = \xi/\sqrt{2} = 0.3320573362\dots,$$

$$c(a = 1, b = 2) = 2\sqrt{2}\xi = 1.3282293448\dots = 2(0.6641146724\dots),$$

$$b(a = 1, c = 1) = \xi^{-2/3} = 1.6551903602\dots$$

$$b(a = 1/2, c = 1) = 2^{1/3}\xi^{-2/3} = 2.0854091764\dots$$

From formulas for the radius of convergence

$$R = \frac{\rho}{(ab)^{1/2}} = \left(\frac{\xi}{ac}\right)^{1/3} \rho$$

and for the limit

$$L = \lim_{x \rightarrow \infty} (bx - z(x)) = \left(\frac{b}{a}\right)^{1/2} \eta,$$

we obtain

$$R(a = 1/2, b = 1) = \sqrt{2}\rho = 5.6900380545\dots,$$

$$R(a = 1, c = 1) = \xi^{1/3}\rho = 3.1273479155\dots,$$

$$R(a = 2, c = 1) = (\xi/2)^{1/3}\rho = 2.4821776854\dots$$

(long ago Weyl [12, 13] gave bounds 2.08 and 3.11 for the latter) and

$$L(a = 1, b = 2) = L(a = 1/2, b = 1) = \sqrt{2}\eta = 1.7207876575\dots,$$

$$L(a = 2, b = 1) = L(a = 1, b = 1/2) = \eta/\sqrt{2} = 0.8603938287\dots$$

When moving fluid encounters a solid, a layer is formed adjacent to the boundary of the solid. Strong frictional effects exist inside this layer; on the outside, by contrast, the flow essentially displays no viscosity [2, 14]. For the case of a thin plate, the fluid velocity changes rapidly from zero (along the plate) to its original value (beyond the boundary layer). Three relevant quantities in this physical model are the *displacement thickness*

$$\delta_1 = \int_0^{\infty} (1 - y'(x)) dx = \eta = 1.2167806216\dots,$$

the *momentum thickness*

$$\delta_2 = \int_0^{\infty} y'(x) (1 - y'(x)) dx = \xi = 0.4695999883\dots$$

and the *energy thickness*

$$\delta_3 = \int_0^{\infty} y'(x) (1 - y'(x)^2) dx = 2 \int_0^{\infty} y''(x) y'(x) y(x) dx = 0.73848498\dots$$

It is also known that [2, 10]

$$y''(x) \sim \kappa \exp \left[ -(x - \eta)^2 / 2 \right]$$

as  $x \rightarrow \infty$ , where  $\kappa = 0.3305407719\dots = (0.2337276212\dots)\sqrt{2}$ . We wonder whether  $\delta_3$  and  $\kappa$  are closely related. The literature associated with  $y(x)$  is massive [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54].

**0.1. Falkner-Skan Equation.** Consider

$$y'''(x) + y''(x)y(x) + \lambda(1 - y'(x)^2) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad \lim_{x \rightarrow \infty} y'(x) = 1$$

which arises in the study of viscous flow past a wedge of angle  $\lambda\pi$ ,  $0 \leq \lambda \leq 1$ . The special case  $\lambda = 0$  is Blasius' equation, in which the wedge reduces to a flat plate. The special case  $\lambda = 1/2$  is called Homann's equation; we here have [2, 3, 55, 56, 57]

$$y''(0) = 0.92768003\dots = (1.31193769\dots)/\sqrt{2}, \quad \lim_{x \rightarrow \infty} (x - y(x)) = 0.804548\dots$$

The special case  $\lambda = 1$  is called Hiemenz's equation (corresponding to stagnation flow, for example, past a large disk); we here have [2, 3, 58, 59]

$$y''(0) = 1.23258765\dots, \quad \lim_{x \rightarrow \infty} (x - y(x)) = 0.647900\dots$$

It is known that a smooth solution  $y(x)$  exists and is unique [12, 13, 60, 61, 62, 63, 64, 65] for each  $\lambda$ ,  $0 \leq \lambda \leq 1$ . An especially simple proof for  $\lambda = 0$ , due to Serrin, appears in [66, 67].

Physically relevant solutions also exist for negative  $\lambda$ , more precisely, in the range  $-0.19883768\dots = \mu \leq \lambda < 0$ . (Positive  $\lambda$  corresponds to flow toward the wedge; negative  $\lambda$  corresponds to flow away from the wedge.) By "physically relevant", we mean that a solution  $y(x)$  further satisfies

$$0 < y'(x) < 1 \quad \text{for all } x > 0,$$

$$1 - y'(x) = O(e^{-\gamma x}) \quad \text{as } x \rightarrow \infty$$

for some  $\gamma > 0$ . It follows that  $y''(0) > 0$  when  $\lambda > \mu$  and  $y''(0) = 0$  when  $\lambda = \mu$ . A deeper understanding of the constant  $\mu$  is desired [17, 58, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79]. Again, the associated literature is massive [18, 19, 20, 21, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102].

**0.2. Streamlines.** At each point in the first quadrant of  $(s, t)$ -space, define a velocity vector  $(u, v)$  by

$$u(s, t) = s^m y'(\theta),$$

$$v(s, t) = -\sqrt{\frac{s^{m-1}}{2(2-\lambda)}} ((\lambda-1)\theta y'(\theta) + y(\theta))$$

where

$$m = \frac{\lambda}{2-\lambda}, \quad \theta = t\sqrt{(m+1)s^{m-1}}.$$

The vector field  $(s, t) \mapsto (u, v)$  determines the streamlines for laminar boundary-layer fluid flow past a wedge, as suggested in the Figure, for a specified viscosity coefficient [3, 83].

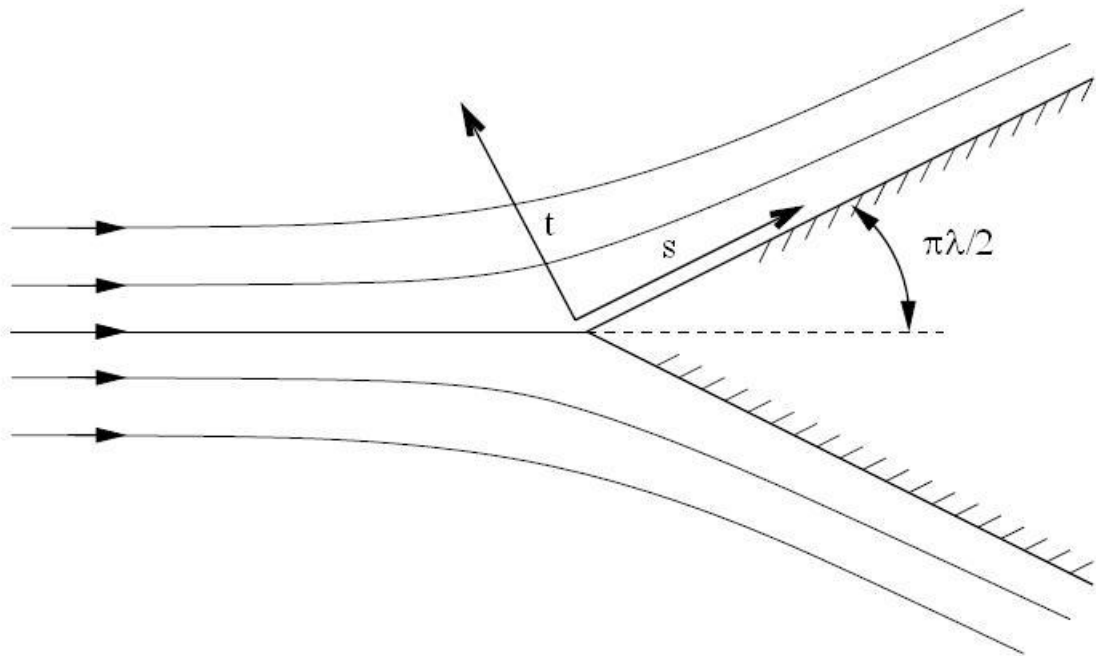


Illustration of Falkner-Skan flow past a wedge.

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