Brachistochrone Problem

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Think of a continuously differentiable curve as a frictionless wire in a vertical plane, with positive x-axis extending to the right and positive y-axis extending downward. Of all curves $y(x)$ joining the origin and a fixed point (p, q) in the first quadrant, which possesses the minimum descent time

$$T = \frac{1}{\sqrt{2g}} \int_0^p \sqrt{\frac{1 + y'(x)^2}{y(x)}} dx$$

from $(0, 0)$ to (p, q)? In words, $y(x)$ is the wire configuration along which a bead will slide, starting from rest, in the shortest possible time T. For simplicity, we take the gravitational acceleration constant g to be $1/2$, so that the coefficient of the integral defining T is 1.

It is well-known that this calculus-of-variations problem reduces to solving the boundary value problem [1, 2, 3, 4, 5, 6]

$$y(x)\left(1 + y'(x)^2\right) = c, \qquad y(0) = 0, \qquad y(p) = q$$

where c is an arbitrary constant, and that $y(x)$ is represented parametrically by

$$x = \frac{c}{2}(t - \sin(t)), \qquad y = \frac{c}{2}(1 - \cos(t)).$$

Let $0 < \theta < 2\pi$ be the unique value satisfying

$$\frac{\theta - \sin(\theta)}{1 - \cos(\theta)} = \frac{p}{q},$$

then

$$T = 2\sqrt{q}\frac{\theta/2}{\sin(\theta/2)}.$$

For example, if $p/q = 1$, then $T/\sqrt{q} = 2.5819045128...$ and if $p/q = \pi/2$, then $T/\sqrt{q} = \pi$. While the latter result is simple, no closed-form expression is known for the former [7, 8].

Interestingly, we have $y'(p) > 0$ when $p/q = 1$, whereas $y'(p) = 0$ when $p/q = \pi/2$. The sliding bead reaches the endpoint with *zero slope* in the latter case.

With this in mind, we introduce a revision of the brachistochrone problem. Let the starting point be $(0, b)$ where $b \geq 0$ is fixed and let the initial speed of the bead along the wire be $\sqrt{2gb}$. Let the endpoint be (p, q), where $p > 0$ is fixed but $q > b$ is free to vary, subject to the constraint that the trajectory slope is zero at (p, q). Of all curves $y(x)$ joining $(0, b)$ and (p, q) satisfying these conditions, which possesses the minimum descent time? [9, 10]

In this revised setting, the boundary value problem is

$$y(x)\,(1 + y'(x)^2) = c, \qquad y(0) = b, \qquad y'(p) = 0$$

and the solution $y(x)$ is represented parametrically by

$$x = -\frac{c}{2}(t + \sin(t)) + p, \qquad y = \frac{c}{2}(1 + \cos(t)).$$

Clearly $q = c$ upon setting $t = 0$. When setting $x = 0$ instead, we obtain

$$p = \frac{c}{2}(t + \sin(t)), \qquad b = \frac{c}{2}(1 + \cos(t))$$

hence

$$t = \arccos\left(\frac{2}{c}b - 1\right)$$

hence

$$\sqrt{(c-b)b} + \frac{c}{2}\arccos\left(\frac{2}{c}b - 1\right) = p$$

hence

$$\sqrt{\left(\frac{q}{p} - \frac{b}{p}\right)\frac{b}{p}} + \frac{1}{2}\frac{q}{p}\arccos\left(2\frac{b/p}{q/p} - 1\right) = 1.$$

Our interest is in the value of q/p, given $b/p = 0$, 1, 2 or 3. If $b/p = 0$, it follows that $q/p = 2/\pi = 0.6366197723...$, consistent with before. If $b/p = 1$, 2 or 3, then

$$q/p = 1.2184055294..., \qquad 2.1201938103..., \qquad 3.0818460494...$$

respectively. The latter value appears in [11, 12], obtained via completely different means.

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