Minkowski-Alkauskas Constant

Steven Finch

May 6, 2008

In addition to examining [1]

$$?\left(0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots\right)=\sum_{k=1}^{\infty}(-1)^{k-1}2^{-(a_1+a_2+\cdots+a_k-1)},$$

we study [2]

$$F\left(a_0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\frac{1|}{|a_3}+\cdots\right)=\sum_{k=1}^{\infty}(-1)^{k-1}2^{-(a_0+a_1+a_2+\cdots+a_k)}.$$

The former is the original Minkowski question mark function, a self-map of $[0,1]$; the latter is defined on the nonnegative real line with $2F(x)=?(x)$ for all $x\in[0,1]$. In particular,

$$F(0)=0,\qquad F(\tfrac{1}{2})=\tfrac{1}{4},\qquad F(1)=\tfrac{1}{2},\qquad F(\sqrt{2})=\tfrac{3}{5},$$

$$F(\tfrac{1+\sqrt{5}}{2})=\tfrac{2}{3},\qquad F(2)=\tfrac{3}{4},\qquad F(3)=\tfrac{7}{8},\qquad \lim_{x\to\infty}F(x)=1^{-}.$$

The distribution F is continuous, strictly increasing, singular, and uniquely determined by the functional equation

$$2F(x)=\begin{cases}F(x-1)+1 & \text{if } x\geq 1,\\ F\left(\dfrac{x}{1-x}\right) & \text{if } 0\leq x<1.\end{cases}$$

Define moments

$$M_\ell=\int_0^\infty x^\ell dF(x),\qquad m_\ell=\int_0^1 x^\ell d?(x)$$

then $m_1=M_1-1=1/2$ follows easily. Similar closed-form expressions for

$$m_2=M_2-4=0.2909264764...,$$

$$m_4=M_4-24m_2-100=0.1269922584...$$

presently do not exist, although progress has recently been made [3]. It is known that

$$2m_3 = 3m_2 - 1/2 = 2(0.1863897146...),$$

$$2M_3 = 9m_2 + 69/2, \qquad 2m_5 = 5m_4 - 5m_2 + 1$$

and analogous relations hold for higher-order moments. Hence calculating m_2, m_4, $\ldots$ to high precision is important for understanding m_3, m_5, $\ldots$.

Alkauskas [4] proved the following asymptotic formula:

$$\begin{aligned} m_\ell &\sim \sqrt[4]{4\pi^2\ln(2)} \cdot c \cdot \left(e^{-2\sqrt{\ln(2)}}\right)^{\sqrt{\ell}} \ell^{1/4} \\ &\sim (2.3562298899...)(0.1891699952...)^{\sqrt{\ell}}\ell^{1/4} \end{aligned}$$

as $\ell \to \infty$, where

$$c = \int_0^1 2^x(1 - F(x))dx = 1.0301995633....$$

This is a fascinating result, especially because m_2, m_4, $\ldots$ remain so mysterious! One would not have expected an asymptotic formula for m_ℓ as such to be possible.

REFERENCES

[1] S. R. Finch, Minkowski-Bower constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 441–443.

[2] G. Alkauskas, The moments of Minkowski ?(x) function: dyadic period functions, arXiv:0801.0051.

[3] G. Alkauskas, The Minkowski question mark function: explicit series for the dyadic period function and moments, arXiv:0805.1717.

[4] G. Alkauskas, Asymptotic formula for the moments of Minkowski question mark function in the interval $[0, 1]$, arXiv:0802.2721.